Moduli spaces and motivic distribution of rational curves

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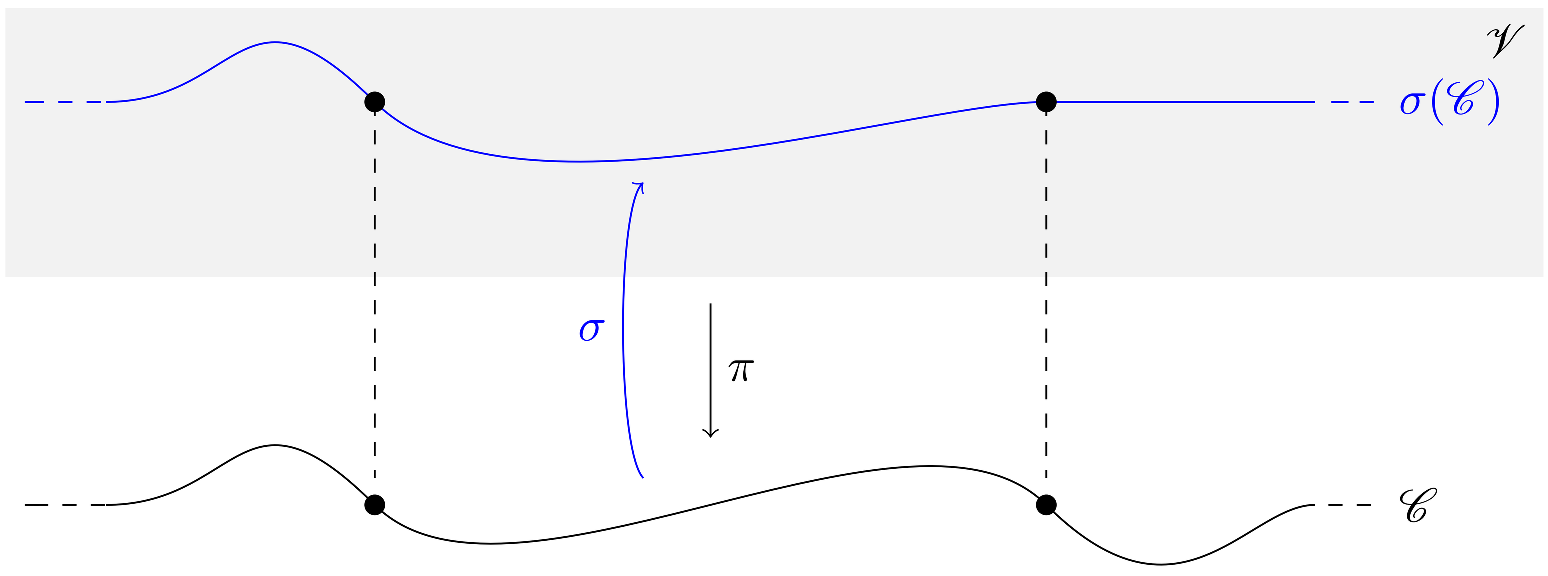
Object of interest

- Given $\pi : \mathscr{V} \to \mathscr{C}$ **proper flat morphism** above a **smooth projective curve** $\mathscr{C}$ over a **field** k whose generic fibre V **is a smooth "Fano-like" variety**
  $\rightsquigarrow$ **Sections** $\sigma : \mathscr{C} \to \mathscr{V}$ of given **numerical class** δ generically intersecting a **given dense open subset** U are parameterised by a quasi-proj. scheme

$$\boxed{\mathrm{Hom}^{\delta}_{\mathscr{C}}(\mathscr{C}, \mathscr{V})_U}$$



$$\boxed{k(\mathscr{C})\text{-points of } V \overset{1:1}{\longleftrightarrow} \text{sections } \sigma : \mathscr{C} \to \mathscr{V}}$$

- Questions
  - ▷ dimension, number of components
  - ▷ if $k = \mathbf{F}_q$ — number of $\mathbf{F}_q$-points (asymptotic formula)
  - ▷ cohomology / motive

Ring(s) of motivic integration

- Grothendieck ring of k-varieties:

$$K_0\mathbf{Var}_k = \frac{\mathbf{Z}\{\text{iso. classes of alg. } k\text{-var.}\}}{\langle [X] - [U] - [X \setminus U] \mid U \underset{\text{open}}{\subset} X \rangle}$$

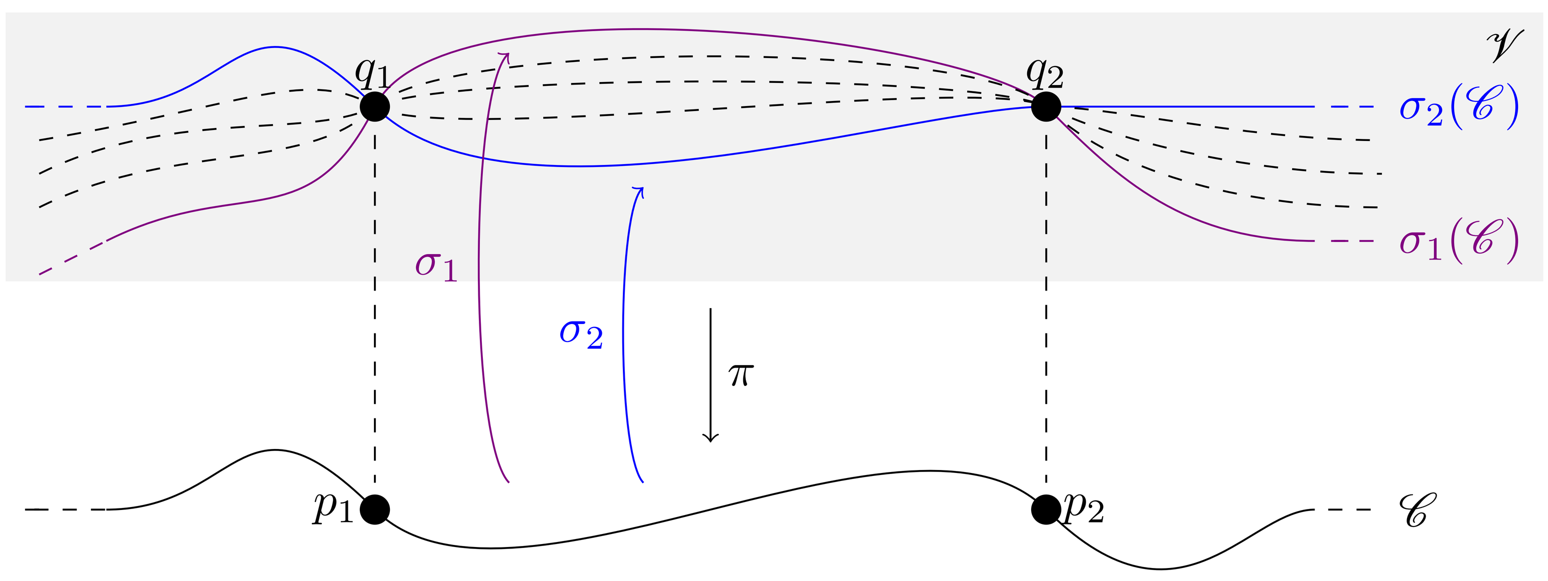
  with product law $[X] \cdot [Y] = [X \times_k Y]$
- Ring of motivic integration:

$$\mathscr{M}_k = K_0\mathbf{Var}_k[\mathbf{L}_k^{-1}]$$

  where $\mathbf{L}_k = [\mathbf{A}^1_k]$
- if $k = \mathbf{F}_q$ — counting measure $[X] \mapsto \#X(\mathbf{F}_q)$

Reduction conditions on curves

Given $\mathscr{S} \underset{\text{closed}}{\hookrightarrow} \mathscr{C}$ and $W \underset{\text{loc. closed}}{\hookrightarrow} \mathrm{Hom}_{\mathscr{C}}(\mathscr{S}, \mathscr{V})$



$$\boxed{\mathrm{Hom}^{\delta}_{\mathscr{C}}(\mathscr{C}, \mathscr{V} \mid W)_U = \{\sigma \mid \sigma_{|\mathscr{S}} \in W\}}$$

Dictionary of global and local fields

$\mathbf{Q}$ *or* Number field K	$\mathbf{F}_q(t)$ *or* Function field of a $\mathbf{F}_q$-curve	$\mathbf{C}(t)$ *or* Function field of a complex curve
$\mathbf{R}, \mathbf{Q}_p, \ldots$ loc. compact	$\mathbf{F}_q((t))$ loc. compact	$\mathbf{C}((t))$ **not** loc. compact

Motivic equidistribution of curves

- Inspired by the Batyrev-Manin-Peyre principle [BM90, Pey95]

$$\#\{\mathbf{x} \in U(\mathbf{Q}) \mid H_{\omega_V^{-1}}(\mathbf{x}) \leqslant B\} \underset{B \to \infty}{\sim} CB \log(B)^{\mathrm{rk}(\mathrm{Pic}(V))-1}$$

  one asks whether

$$\left[\mathrm{Hom}^{\delta}_{\mathscr{C}}(\mathscr{C}, \mathscr{V} \mid W)_U\right] \mathbf{L}_k^{-\delta \cdot \omega_V^{-1}} \underset{\text{“}\delta \to \infty\text{”}}{\longrightarrow} \tau(\mathscr{V} \mid W)$$

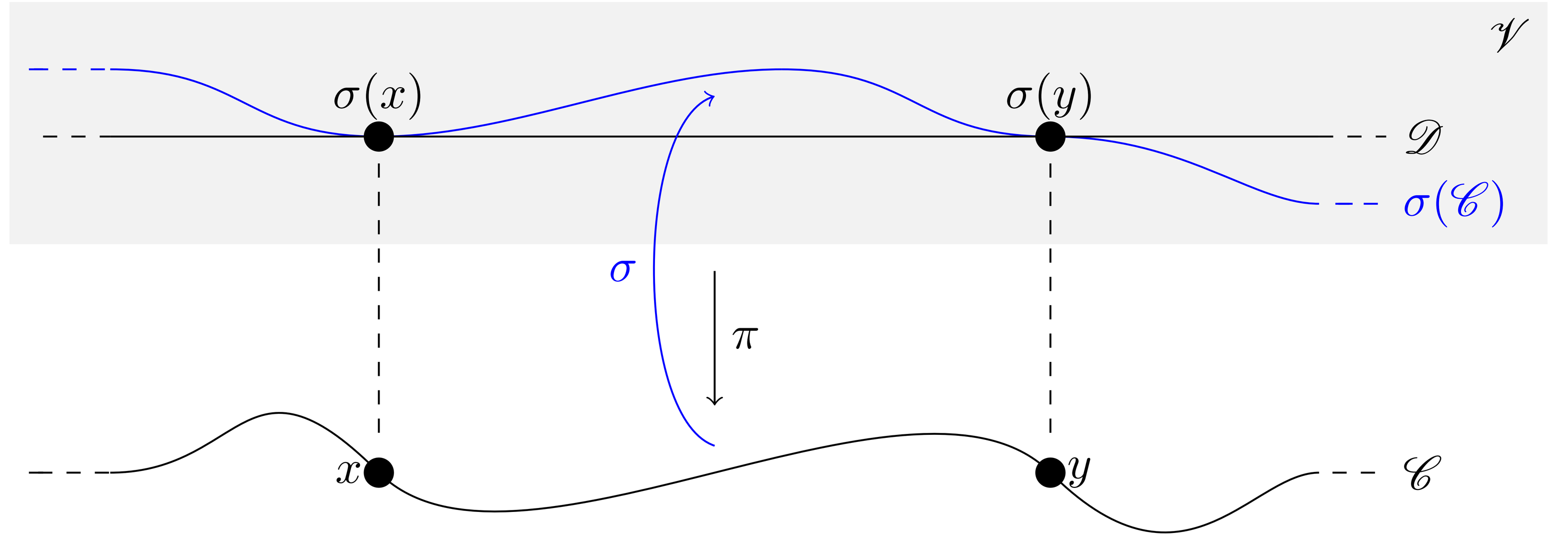
  in a suitable completion of $\mathscr{M}_k$.
- The expected limit $\tau(\mathscr{V} \mid W)$ is given by $\mathbf{L}_k^{\dim(V)(1-g(\mathscr{C}))}$ times the motivic Euler product:

$$\frac{\left[\mathrm{Pic}^0(\mathscr{C})\right]^{\mathrm{rk}(\mathrm{Pic}(V))}}{(\mathbf{L}_k^{g(\mathscr{C})}(1 - \mathbf{L}_k^{-1}))^{\mathrm{rk}(\mathrm{Pic}(V))}} \prod_{p \in \mathscr{C}} (1 - \mathbf{L}_{\kappa(p)}^{-1})^{\mathrm{rk}(\mathrm{Pic}(V))} \int_{W_p} \omega_p$$

  where ω_p a certain local motivic measure [Fai23a].

Tangency conditions at every point

Campana-type condition: if σ locally intersects an effective divisor $\mathscr{D}$, then it does it with multiplicity $\geqslant m$. For example, on the picture below we took $m = 2$.



Equivariant compactifications

- **Theorem** [Fai23a]**.** Rational curves equidistribute on toric varieties.
- A similar result holds for sections with Campana conditions on equivariant compactifications of vector groups [Fai23b].

Questions

- New examples?
- Reformulation in log-geometry? vs. root-stacks and orbifolds?
- Going back to classical countings when $k = \mathbf{F}_q$?

Techniques

- Universal torsors for Mori Dream Spaces
- Motivic harmonic analysis, circle method [BB23], ...
- ...other ones to be developed!

References.

[BB23] Margaret Bilu and Tim Browning, *A motivic circle method*, preprint, arXiv:2304.09645 (2023).

[BM90] Victor V. Batyrev and Yuri I. Manin, *Sur le nombre des points rationnels de hauteur borné des variétés algébriques*, Mathematische Annalen **286** (1990), no. 1-3, 27–43.

[Fai23a] Loïs Faisant, *Motivic distribution of rational curves and twisted products of toric varieties*, preprint, arXiv:2302.07339 (2023).

[Fai23b] ______, *Stabilisation phenomena in moduli spaces of curves*, PhD thesis, Université Grenoble Alpes, 2023.

[Pey95] Emmanuel Peyre, *Hauteurs et mesures de Tamagawa sur les variétés de Fano*, Duke Mathematical Journal **79** (1995), no. 1, 101–218.