Some applications of p-adic integration to geometry and arithmetics

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These notes are lecture notes for a graduate course given at the Institute of Science and Technology Austria in Spring 2025.

Main references and prerequisites. We assume familiarity with basic notions of commutative algebra, algebraic geometry and differential geometry.

A very classical reference concerning p-adic integration is Igusa's monograph [**Igu00**]. The proof of Batyrev's theorem closely follows the original article [**Bat99**] - see also [**CLNS18**, Ch. 1].

Likewise, the proof of Denef's formula follows [**Den87**, **VZG08**] and [**CLNS18**, Ch. 1]. For a recent survey on the monodromy conjecture, see [**Vey24**].

Concerning number theory, we assume familiarity with the geometrical point of view adopted for example in the first chapters of Neukirch's book [**Neu13**] (and for German speakers, its original version [**Neu06**]).

Concerning more specifically Fourier analysis on number fields and Adèles, most of the basics covered by these notes can be found in the book of Dinakar Ramakrishnan and Robert J. Valenza [**RV13**]. Of course, Tate's thesis [**Tat67**] is a main source for all this, which is also covered by [**Lan94**, Chap. XIV].

Contents

Introduction

First, let us say that this course has four guiding results.

1. Topology: p-adic integration towards Batyrev's theorem

The first main result of these notes has a topological flavor and is known as Batyrev's theorem for Calabi-Yau (complex) varieties.

THEOREM 1.1 ([Bat99]). *Let X and Y be two Calabi-Yau varieties, which are birational. Then X and Y have the same Betti numbers.*

The proof we present in these notes is the original one which makes use of p-adic integration combined with Deligne's proof of Weil's conjectures. Actually, using motivic integration, it is possible to prove a stronger statement, namely equality of Hodge numbers (and even isomorphism of Hodge structures); but this generalisation, due to Kontsevich (talk in Orsay, 1995), is out of the scope of these notes.

2. Singularity theory: Denef's formula and the Monodromy conjecture

Moving on to the interplay with singularity theory, the second main result we present is known as Denef's formula for Igusa's zeta function.

Let R be the valuation ring of a non-Archimedean local field F of characteristic zero. For all $\boldsymbol{y} \in K^r$, let $\|\boldsymbol{y}\| = \max_{i=1}^r(|y_i|)$ where $|\cdot|$ is the absolute value of F.

DEFINITION 2.1. Let $f = (f_1, \dots, f_r)$, where $f_i \in R[x_1, \dots, x_m]$. The local zeta function associated to f is the complex function defined by:

$$Z_f(s) = \int_{R^m} \|f(\boldsymbol{x})\|^s \mathrm{d}\boldsymbol{x}, \ s \in \mathbf{C}.$$

Denef's formula is an explicit formula for Igusa's zeta function in terms of resolutions of singularities.

THEOREM 2.1. *Let $\mathbf{K}$ be a number field. Let $X = V(I) = V(f_1, \dots, f_r) \subseteq \mathbb{A}^m_{\mathbf{K}}$ and $h : Y \to X$ be an embedded resolution of singularities, with numerical data $(N_i, \nu_i)_{1 \leq i \leq t}$. Let $\mathfrak{p} \subseteq \mathcal{O}_{\mathbf{K}}$ be a non-zero prime ideal with residue field k. Suppose that h has good reduction modulo $\mathfrak{p}$. Then:*

$$Z_f(s) = q^{-m} \cdot \sum_{I \subseteq \{1, \dots, t\}} c_I \cdot \prod_{i \in I} \frac{(q-1)q^{-N_i s - \nu_i}}{1 - q^{-N_i s - \nu_i}},$$

where $c_I = \sharp\{a \in Y(k) \mid a \in E_i(k) \Leftrightarrow i \in I\}$ and $q = \sharp k$.

We will also explain how this is related to an important conjecture concerning the monodromy.

3. Arithmetic geometry: an Adelic Poisson formula in use

Let K be any global field: either a number field, or the field of function of a curve over a finite field. Let $\mathbb{A}_K$ be its space of Adèles. It is possible to use p-adic integration to define a theory of integration on $\mathbb{A}_K$.

THEOREM 3.1 (Poisson formula). *Let f be a continuous fuction on $\mathbb{A}_K$ such that both f and $\widehat{f}$ are integrable and that*

$$\sum_{\gamma\in K} f(\gamma+x)$$

converges absolutely and uniformly when x belongs to any compact subset of $\mathbb{A}_K/K$.

Then

$$\sum_{\gamma\in K} f(\gamma+x) = \sum_{\gamma\in K} \widehat{f}(\gamma+x)$$

for every adèle $x\in\mathbb{A}_K$. In particular,

$$\sum_{\gamma\in K} f(\gamma) = \sum_{\gamma\in K} \widehat{f}(\gamma).$$

3.1. Tamagawa numbers of connected semi-simple algebraic groups. Finally heading towards arithmetic geometry, the third result is an application of the adelic Poisson formula.

It concerns an invariant of Adelic nature for algebraic groups defined over a global field K: the Tamagawa number τ_K.

THEOREM 3.2. *For any positive integer n,*

$$\tau_{\mathbf{Q}}(\mathrm{SL}_n) = \int_{\mathrm{SL}_n(\mathbf{Q})\backslash \mathrm{SL}_n(\mathbb{A}_{\mathbf{Q}})} \mathrm{d}\mu = 1$$

This theorem is a special case of the following conjecture.

Conjecture 3.1 (Weil). *Let G be a connected, simply connected semi-simple algebraic group over a global field, then its Tamagawa number equals one.*

3.2. Counting points on certain blow-ups of the projective plane. The following is due to Chambert-Loir and Tschinkel [CLT00].

THEOREM 3.3. *Let U be the complement in $\mathbf{P}^2_{\mathbf{Q}}$ of $\{x_0=0\}\simeq\mathbf{P}^1_{\mathbf{Q}}$ and let $p_1,\dots,p_r$ distinct $\mathbf{Q}$-points on this latter line.*

Let X be the blow-up of $\mathbf{P}^2_{\mathbf{Q}}$ at the r points $p_1,\dots,p_r$ and $H_{\omega_X^{-1}}$ be the exponential height associated to the metrized line bundle ω_X^{-1}. We identify U with its preimage in X.

Then for every real number $B>0$ the set

$$\{x\in U(\mathbf{Q}) \mid H_{\omega_X^{-1}}\leqslant B\}$$

is finite and

$$\#\{x\in U(\mathbf{Q}) \mid H_{\omega_X^{-1}}\leqslant B\} \sim \frac{1}{3\cdot 2^r\cdot r!}\tau(\omega_X)B\cdot\log(B)^r$$

as $B\to\infty$.

Remark: it is possible to state this result for arbitrary number fields.

CHAPTER 1

Basics of p-adic integration towards Batyrev's theorem

ABSTRACT. The guiding thread of this chapter is a theorem, whose proof using p-adic integration is due to Batyrev, stating that two birationally equivalent Calabi-Yau varieties share the same Betti numbers.

We start with recalling general definitions and facts about local and global fields, before coming to a general definition of p-adic integrals on non-Archimedean analytic manifolds. We conclude the chapter with the proof of Batyrev's theorem.

1. Local fields

1.1. First definitions. We start with a few definitions and examples.

DEFINITION 1.1. An *absolute value* on K is a map

$$| \cdot | : K \to \mathbf{R}^+$$

such that

- $(|0|, |1|) = (0, 1)$,
- $|a + b| \leqslant |a| + |b|$ for all $a, b \in K$,
- $|ab| = |a||b|$ for all $a, b \in K$.

The datum of a field together with an absolute value is called *a valued field*.

DEFINITION 1.2. A *local field* is a valued field

- whose absolute value is non-trivial (!)
- and whose associated metric topology is locally compact.

Equivalently and more concretely, a local field is either $\mathbf{R}$, $\mathbf{C}$, a finite extension of the field $\mathbf{Q}_p$ of p-adic numbers, or a field of formal Laurent series $\mathbf{F}_q((t))$, for some power q of a prime p.

EXAMPLE 1.1. Given a prime p, one can define on $\mathbf{Q}$ the p-adic absolute value by

$$|a|_p = p^{-\nu_p(a)}$$

where $\nu_p(a)$ (for $a \neq 0$) is the p-adic valuation of a, that it is to say, the unique integer ν such that $a = p^\nu m/n$ with $\gcd(p, m) = \gcd(p, n) = 1$.

EXAMPLE 1.2. In a similar way, one can define a t-adic absolute value on $\mathbf{F}_q(t)$.

PROPOSITION 1.1. *Every valued field K admits a completion $\widehat{K}$ containing K as a dense subset on which the absolute value of $\widehat{K}$ coincides with the original one of K.*

REMARK 1.1. A local field is automatically complete.

EXAMPLE 1.3. The completion of $\mathbf{Q}$ with respect to the p-adic absolute value is the field of p-adic numbers. The closed unit disk in $\mathbf{Q}_p$ is the ring of p-adic integers.

1.2. Non-Archimedean valued fields.

Definition 1.3. A valued field K is said to be *non-Archimedean* if its absolute value $|\cdot|$ satisfies the ultrametric inequality

$$|x+y| \leqslant \max(|x|,|y|) \quad x,y \in K.$$

Definition 1.4. Let K be a non-Archimedean valued field. The closed unit disk

$$R = \{x \in K \mid |x| \leqslant 1\}$$

is a subgring of K, called the *valuation ring of* K. This terminology means that for any $x \in K^\times$, either x or x^{-1} lies in R.

The ring R is a local ring whose unique maximal ideal is the open unit disk

$$\mathfrak{m} = \{x \in K \mid |x| < 1\}.$$

The *residue field of* K is the quotient of R by $\mathfrak{m}$. In these notes it will generally be denoted by the greek letter κ.

Example 1.4. In $\mathbf{F}_q(t)$, respectively $\mathbf{F}_q((t))$, both endowed with the t-adic absolute value

$$|x| = q^{-v_t(x)},$$

the valuation ring is the polynomial ring $\mathbf{F}_q[t]$, respectively the ring of power series $\mathbf{F}_q[[t]]$.

1.3. Hensel's lemma. Henselianity is a crucial property that we are going to use a lot later, without really thinking about it.

Definition 1.5. A local ring R, with maximal ideal $\mathfrak{m}$ and residue field k, is called *henselian* if for every polynomial $f \in R[T]$ and every $a \in R$ such that

$$f(a) \in \mathfrak{m} \text{ and } f'(a) \notin \mathfrak{m}$$

there *exists* a *unique* $b \in R$ such that $f(b) = 0$ and $b - a \in \mathfrak{m}$.

Geometrically, it means that for every $f \in R[T]$, any smooth k-point of the special fibre of the R-suhscheme of $\mathbf{A}^1_R$ defined by f lifts to a unique R-point of $\{f = 0\}$.

Lemma 1.1 (Hensel's lemma, one variable). *Let R be a complete discrete valuation ring and $f \in R[T]$. Assume that there exists integers $n \geqslant e \geqslant 0$ and $a \in R$ such that*

$$f(a) \in \mathfrak{m}^{n+e+1} \text{ and } f'(a) \notin \mathfrak{m}^{e+1}.$$

Then there exists a unique lift $b \in R$ of a modulo $\mathfrak{m}^{n+1}$, that is to say such that

$$f(b) = 0 \text{ and } b - a \in \mathfrak{m}^{n+1}.$$

Remark 1.2. The previous lemma says that in particular, complete discrete valuation rings are henselian.

Proof. This is a non-Archimedean application of Newton's algorithm. One has to find a solution of the equation in u

$$f(a + \varpi^{n+1}u) = 0$$

where ϖ is a generator of $\mathfrak{m}$. We see a as the initial guess for a root of f and we must define the next terms of Newton's iteration and control their size. In what follows, when we write $x \in \mathfrak{m}^m$ for some integer m, just think about the non-Archimedean valuation being at least equal to m.

First, applying the Taylor expansion of f

$$f(a+T) = f(a) + f'(a)T + T^2 \underbrace{g(T)}_{\in R[T]}$$

to our equation gives

$$f(a) + f'(a)\varpi^{n+1}u + \varpi^{2n+2}u^2 g(\varpi^{n+1}u) = 0.$$

Then, by assumption, one can write

$$f(a) = u_1 \varpi^{n+e+1}$$

where $u_1 \in R$ and the ratio

$$u_2 = \frac{\varpi^e}{f'(a)}$$

actually lies in R, since $f'(a)$ is not divisible by ϖ^{e+1}. Dividing everywhere by $f'(a)\varpi^{n+1}$ our equation becomes

$$\begin{aligned} u &= -\frac{f(a)}{f'(a)\varpi^{n+1}} - \frac{\varpi^{2n+2}}{f'(a)\varpi^{n+1}} u^2 g(\varpi^{n+1}u) \\ &= -u_1u_2 - \varpi^{n+1-e} u_2 u^2 g(\varpi^{n+1}u). \end{aligned}$$

Since we assume that $n \geqslant e$, we have that necessarily $u + u_1u_2 \in \mathfrak{m}$. Now one puts

$$\begin{aligned} a_1 &= a - \varpi^{n+1}u_1u_2 \\ &= a - \frac{f(a)}{f'(a)}. \end{aligned}$$

The reduction classes of $a_0 = a$ and a_1 agree modulo $\mathfrak{m}^{n+1}$ and $f(a_1) \in \mathfrak{m}^{n+e+1+1}$, getting one first step closer to a root of f. Moreover, it is important to check that $f'(a_1) \notin \mathfrak{m}^{e+1}$ by using the Taylor expansion of f' and the fact that $n \geqslant e$:

$$f'(a_1) = \underbrace{f'(a)}_{\notin \mathfrak{m}^{e+1}} + \underbrace{f''(a)(a_1 - a)}_{\in \mathfrak{m}^{n+1}} + \underbrace{(a_1 - a)^2 h(a_1 - a)}_{\in \mathfrak{m}^{2n+2}} \qquad \text{hence } f'(a_1) \notin \mathfrak{m}^{e+1}.$$

This a_1 we've just constructed is unique modulo $\mathfrak{m}^{n+2}$. Indeed, if a' is another element of R satisfying $a' - a_0 \in \mathfrak{m}^{n+1}$ and $f(a') \in \mathfrak{m}^{n+e+1+1}$, then $a' - a_1 \in \mathfrak{m}^{n+1+1}$. This can be seen by writing

$$f(a') = f(a_1) + f'(a_1)(a' - a) + (a' - a_1)^2 g(a' - a_1)$$

and then the fact that $f'(a_1) \notin \mathfrak{m}^{e+1}$ forces $a' - a_1 \in \mathfrak{m}^{n+1+1}$.

The Newton sequence $(a_m)_{m \in \mathbf{N}}$ is now defined by setting

$$\begin{aligned} a_0 &= a \\ a_{m+1} &= a_m - \frac{f(a_m)}{f'(a_m)} \quad m \in \mathbf{N}. \end{aligned}$$

Repeating the previous argument, one gets that $a_{m+1} - a_m \in \mathfrak{m}^{n+m+1}$ for every $m \in \mathbf{N}$ and $f(a_m) \in \mathfrak{m}^{n+m+e+1}$. By completeness of R, this sequence converges to an element $b \in R$ such that

$$f(b) = 0 \text{ and } b - a \in \mathfrak{m}^{n+1}.$$

This lift b is unique: indeed, if $b' \in R$ is another element such that $f(b') = 0$ and $b' - a \in \mathfrak{m}^{n+1}$, then necessarily $b' - a_m \in \mathfrak{m}^{n+m+1}$ for every $m \in \mathbf{N}$ (by induction on m, and we already did the case $m = 1$ above), so that $b' = b$. □

With a little more effort, one can show a multivariate version of Hensel's lemma with formal power series. The proof is left as an exercise for very brave students.

LEMMA 1.2 (Hensel's lemma). *Let R be a complete discrete valuation ring and $\mathfrak{m}$ its maximal ideal. Fix integers $r \geq \ell \geq 0$. Consider $f_1, \dots, f_\ell \in R[\![T_1, \dots, T_r]\!]$ and $a_1, \dots, a_r \in \mathfrak{m}$ such that*

$$f_i(a) \in \mathfrak{m} \text{ for all } i \in \{1, ..., \ell\}.$$

Assume moreover that the minor

$$\Delta = \det \left(\frac{\partial f_i}{\partial T_j} \right)_{1 \leq i,j \leq \ell}.$$

of the Jacobian matrix is invertible in R.

Then there exist $b_1, \dots, b_r \in R$ such that

$$f_i(b) = 0 \text{ for all } i \in \{1, ..., \ell\}$$

and

$$a_j \equiv b_j \bmod \mathfrak{m} \text{ for all } j \in \{1, ..., r\}.$$

2. Analytic manifolds and integration

We work above a local field K, which can be Archimedean or not. The goal of this section is to define K-analytic manifolds and integration on them.

In practice, in this course we will consider manifolds that are realised as the set

$$M = X(K)$$

of K-point of a smooth K-scheme X of finite type and pure dimension. To fix ideas, the reader can think about $K = \mathbf{Q}_p$ or $K = \mathbf{F}_q((t))$ but the definitions given above also apply to $K = \mathbf{R}$ or $\mathbf{C}$.

2.1. Analytic functions.

In this subsection it is sufficient to assume that K is a complete valued field in the sense of Definition 1.1 page 7.

Let $T_1, ..., T_d$ be indeterminates. We use the convenient notation $\mathbf{T}^{\boldsymbol{m}} = T_1^{m_1} ... T_d^{m_d}$ for every $\boldsymbol{m} \in \mathbf{N}^d$.

DEFINITION 2.1. A *convergent* power series in d variable is an element

$$f(\mathbf{T}) \in K[[\mathbf{T}]] = K[[T_1, ..., T_d]]$$

such that the radius of convergence

$$\rho(f) = \sup \left\{ r \in \mathbf{R}_+ \,\middle|\, \text{the sequence } \left(|f_{\boldsymbol{n}}| r^{|\boldsymbol{n}|} \right)_{\boldsymbol{n} \in \mathbf{N}^d} \text{ converges to } 0 \right\}$$

is positive.

Analytic functions are precisely the functions coming locally from a convergent power series.

DEFINITION 2.2. Let U be an open subset of K^d. A function

$$f : U \to K$$

is said to be K-analytic if for every point $a \in U$ there exists a convergent power series

$$f_a \in K[[\mathbf{T}]]$$

such that

$$f(x) = f_a(x - a)$$

for all $x \in D_a(a, \rho(f_a))$.

REMARK 2.1.
- From the definition, one sees that a K-analytic function is automatically continuous.
- Hence, inside the sheaf of continuous functions on U taking values in K, one can define the subsheaf of K-analytic functions.
- The set of K-analytic functions on U admits a structure of K-algebras.

THEOREM 2.1 (Implicit function theorem). *Let K be a complete valued field. Let m and n be two positive integers and let $\mathbf{X} = (X_1, \dots, X_m)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ be two sets of indeterminates.*

Assume that we are given n formal power series

$$F_1, \dots, F_n \in K[\![\mathbf{X}, \mathbf{Y}]\!]$$

such that

$$F_i(0,0) = 0 \textit{ for all } i \in \{1, ..., n\}$$

and

$$\det\left(\frac{\partial F_i}{\partial Y_j}(0,0)\right)_{1 \le i,j \le n} \neq 0.$$

Then, there exists a unique n-tuple of elements

$$f_1, \dots, f_n \in K[\![\mathbf{X}]\!]$$

such that

$$F_i(\mathbf{X}, f_1(\mathbf{X}), ..., f_n(\mathbf{X})) = 0$$

for all $i \in \{1, ..., n\}$.

***In case** F **converges** on a neighbourhood of $\mathbf{0} \in K^{m+n}$, **then** f **converges as well** on a neighbourhood of $\mathbf{0} \in K^m$, and on a small enough neighbourhood of $\mathbf{0} \in K^{m+n}$, the vanishing locus of F coincides with the graph of f.*

Moreover, if the valuation on K is discrete and R is the valuation ring of K, the F_i's are all in $R[\![\mathbf{X}, \mathbf{Y}]\!]$ and

$$\det\left(\frac{\partial F_i}{\partial Y_j}(0,0)\right)_{1 \le i,j \le n} \notin \mathfrak{m},$$

then

$$f_1, \dots, f_n \in R[\![\mathbf{X}]\!].$$

PROOF. After an initial simplification, the proof is still as brutal as one can imagine.

First, one remarks that our problem is invariant under base change for the coefficients of the F_i's. Write

$$F_i(\mathbf{X},\mathbf{Y}) = \sum_{j=1}^{n} a_{ij} Y_j - \underbrace{\sum_{|\boldsymbol{j}|+|\boldsymbol{k}|>0} c_{i\boldsymbol{j}\boldsymbol{k}} \mathbf{X}^{\boldsymbol{i}} \mathbf{Y}^{\boldsymbol{k}}}_{=G_i(\mathbf{X},\mathbf{Y})}$$

so that

$$A = (a_{ij})_{i,j\in\{1,\dots,n\}} = \left(\frac{\partial F_i}{\partial Y_j}(0,0)\right)_{i,j\in\{1,\dots,n\}}$$

is invertible by assumption. Multiplying by A^{-1} the coefficients of $(F_1,\dots,F_n)$, we can assume that A is the identity matrix without changing our problem.

Now we have to solve the system

$$f_i(\mathbf{X}) = \sum_{|\boldsymbol{j}|+|\boldsymbol{k}|>0} c_{i\boldsymbol{j}\boldsymbol{k}} \mathbf{X}^{\boldsymbol{j}} (f_1(\mathbf{X}),\dots,f_n(\mathbf{X}))^{\boldsymbol{k}} \qquad i\in\{1,\dots,n\}.$$

Let us write

$$f_i(\mathbf{X}) = \sum_{d\in\mathbf{N}} f_{i,d}(\mathbf{X})$$

where $f_{i,d}(\mathbf{X})$ is the homogeneous part of degree d of $f_i(\mathbf{X})$, so that we want to check that the $f_{i,d}(\mathbf{X})$'s are uniquely determined. The previous system is equivalent to

$$f_{i,d}(\mathbf{X}) = \sum_{\substack{|\boldsymbol{j}|+|\boldsymbol{k}|>0 \\ k_1+\dots+k_n=|\boldsymbol{k}| \\ d_{i',k'}\in\mathbf{Z}_{>0} \\ |\boldsymbol{j}|+\sum_{i',k'} d_{i',k'}=d}} c_{i\boldsymbol{j}\boldsymbol{k}} \mathbf{X}^{\boldsymbol{j}} \prod_{i'\in\{1,\dots,n\}} \prod_{k'\in\{1,\dots,k_i\}} f_{i',d_{i',k'}}(\mathbf{X})$$

In particular

$$f_{i,1}(\mathbf{X}) = \sum_{|\boldsymbol{j}|=1} c_{i\boldsymbol{j}0} \mathbf{X}^{\boldsymbol{j}}.$$

To be completed: one eventually shows that all the other coefficients are uniquely determined. □

Partial derivatives of K-analytic functions $f : U \to K$ are defined as usual by the formula:

$$\frac{\partial f}{\partial x_i}(a) = \lim_{t\to 0} \frac{f(a+t\epsilon_i)-f(a)}{t}$$

where $\varepsilon_1,\dots,\varepsilon_d$ is the canonical basis of K^d. They are automatically K-analytic, because they coincide locally with the formal derivatives of the convergent power series defining f in a neighborhood of $a\in U$.

Definition 2.3. Let $f : U \to K^d$ be a K-analytic function on an open subset $U \subset K^d$. The Jacobian determinant of f is defined at $a\in U$ by

$$\mathrm{Jac}(f)(a) = \det\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{1\le i,j\le n}.$$

Applying Theorem 2.1, we are able to locally invert K-analytic functions whose Jacobian does not vanish in a neighborhood of a point.

THEOREM 2.2 (Local inversion). *Let $f : U \to K^d$ be a K-analytic function on an open subset $U \subset K^d$. Let $a \in U$ such that*

$$\mathrm{Jac}(f)(a) \neq 0.$$

Then, there exist

- *an open neighborhood U_a of a such that $f(U_a)$ is also an open neighborhood of $f(a)$ in K^d ;*
- *a K-analytic function*
$$g_a : f(U_a) \to U_a$$
such that
$$g \circ f = \mathrm{id}_{U_a} \text{ and } f \circ g = \mathrm{id}_{f(U_a)}.$$

2.2. Analytic manifolds.

DEFINITION 2.4. A K-analytic manifold of dimension d can be defined in two ways.

(1) (Concrete) It is a topological space M together with a d-dimensional K-analytic atlas on it: a set of mutually compatible charts (U_i, φ_i) such that the union of the sets U_i covers M (called an atlas), where compatible means that for all i, j, the homeomorphism
$$\varphi_i(U_i \cap U_j) \xrightarrow{\varphi_j \circ \varphi_i^{-1}} \varphi_j(U_i \cap U_j)$$
is K-analytic.

(2) (Abstract) It is a locally K-ringed space $(M, \mathscr{O}_M)$ which is locally isomorphic to the polydisk
$$E^d(0,1) = \{x \in K^d \mid |x_i| \leqslant 1 \text{ for all } i \in \{1, ..., d\}\}$$
endowed with its sheaf of K-analytic functions.

2.3. Change of variables and gauge forms. From now on we assume that K is a local field in the sense of Definition 1.2.

DEFINITION 2.5. Assume that μ is a Haar measure on $(K, +)$. The locally compact group $(K^d, +)$ can be endowed with an induced Haar measure

$$\mathrm{d}\mu(\boldsymbol{x}) = \mathrm{d}\mu(x_1) \otimes ... \otimes \mathrm{d}\mu(x_d).$$

DEFINITION 2.6 (Modulus of $(K, +, \mu)$). The modulus

$$\mathrm{mod}_K : K \to \mathbf{R}_+$$

is defined by the formula

$$\mu(a\Omega) = \mathrm{mod}_K(a)\mu(\Omega)$$

for every $a \in K$ and every bounded measurable subset Ω of K.

Definition 2.7. Let U be an open subset of K^d and ω be a differential form of degree d on U. In other words, there exists a unique analytic function h on U such that

$$\omega = h(\boldsymbol{x})\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d.$$

The measure

$$\mathrm{mod}_K(\omega)$$

is defined by

$$\int_U \varphi \,\mathrm{mod}_K(\omega) = \int_U \varphi(x)\mathrm{mod}_K(h(\boldsymbol{x}))\mathrm{d}\mu(\boldsymbol{x})$$

for every continuous function φ on U having compact support.

Theorem 2.3 (Local change of variables). *Let U be an open set in K^d and let*

$$f : U \to K^d$$

be an injective K-analytic map. Assume moreover that the Jacobian of f does not vanish on U.

Then, for every integrable function $\varphi : f(U) \to \mathbf{R}$

$$\int_{f(U)} \varphi(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) = \int_U \varphi(f(\boldsymbol{x}))\mathrm{mod}_K(\mathrm{Jac}(f)(\boldsymbol{x}))\mathrm{d}\mu(\boldsymbol{x}).$$

Proof. It is enough to prove the formula on a small open neighborhood of every point of U, prove it for some elementary functions, and then use the chain rule of the Jacobian.

Let ϖ be a generator of $\mathfrak{m}$.

(1) First, one proves it for a linear change of variable $y = Ax + a$ where $A \in \mathrm{GL}_d(K)$ and $a \in K^d$. This comes from the formula

$$\mu(A\Omega) = \mathrm{mod}_K(\det(A))\mu(\Omega).$$

Remark: in most concrete situations this is already enough!

(2) Then one proves it for *special restricted power series*, that is to say for $f \in K[\![\mathbf{X}]\!]$ such that

$$f(0) = 0$$

and $c_{\boldsymbol{i}} \in \mathfrak{m}^{|\boldsymbol{i}|-1}$ for every $\boldsymbol{i} \in \mathbf{N}^d \setminus \{\mathbf{0}\}$, in particular $f \in R[\![\mathbf{X}]\!]$. In that case, $y = f(x)$ is measure-preserving. Indeed, the image of $a + \varpi^e R^d$ under f is $f(a) + \varpi^e R^d$ for every $e \in \mathbf{Z}_{>0}$.

(3) If $f \in R[\![\boldsymbol{X}]\!]$ is convergent in a neighborhood of some a, then for every $e \in \mathbf{Z}_{>0}$

$$g(\mathbf{X}) = \varpi^{-e}(f(a + \varpi^e\mathbf{X}) - f(a))$$

is a special restricted power series.

(4) Hence one can always assume that $f(a) = \mathbf{0}$ and that $f_i(\mathbf{X})$ is of the form

$$f_i(\mathbf{X}) = X_i + \sum_{|\boldsymbol{j}|>2} c_{i,\boldsymbol{j}}\mathbf{X}^{\boldsymbol{j}}$$

and since f_i converges locally, there are some well-chosen $e_0, e_1 \in \mathbf{N}$ such that $\varpi^{e_1}c_{i,\boldsymbol{j}}\varpi^{e_0|\boldsymbol{j}|}$ is in R for every $i \in \{1, \ldots, d\}$ and $\boldsymbol{j}$ (choose e_0 such that $c_{i,\boldsymbol{j}}\varpi^{e_0|\boldsymbol{j}|} \to 0$ as $|\boldsymbol{j}| \to \infty$ for every i and then choose e_1). One can form a special restricted power series $g_i(\mathbf{X})$

$$g_i(\mathbf{X}) = \varpi^{-e}f_i(\varpi^e\mathbf{X})$$

for $e \geqslant 2e_0 + e_1 + 1$.

Then one concludes arguing by composition. □

The measure associated to a top-degree global differential form – also called *gauge form* – on an analytic manifold is defined locally using charts. Naturally, it depends on the existence of such a global form and on the choice of the differential form.

PROPOSITION 2.1. *If M is a K-analytic manifold of dimension d and ω is a differential form of degree d on M, then there exists a unique measure*

$$\mathrm{mod}_K(\omega)$$

on M which locally coincides with the measure associated to differential forms on open subsets of K^d, that is to say such that for every chart (U, f) of M and every integrable function φ having support in U,

$$\int_M \varphi \, \mathrm{mod}_K(\omega) = \int_{f(U)} (\varphi \circ f^{-1}) \, \mathrm{mod}_K((f^{-1})^*\omega).$$

PROOF. First, assume that $M = U$ is a open subset of K^d. Then,

$$\omega = h\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$$

for a unique analytic function h on U. As we already saw, the measure $\mathrm{mod}_K(\omega)$ is then given by

$$\int_U \varphi \, \mathrm{mod}_K(\omega) = \int_U \varphi(x)\mathrm{mod}_K(h(x))\mathrm{d}\mu(x)$$

for every compactly supported continuous function on U. This definition is invariant by K-analytic differomorphism: if $g : V \to U$ is such a change of coordinates,

$$\begin{aligned}\int_U \varphi \, \mathrm{mod}_K(h(x))\mathrm{d}\mu(x) &= \int_V \varphi \circ g(y) \, \mathrm{mod}_K(h \circ g(y)) \, \mathrm{mod}_K(\mathrm{Jac}(g)(y))\mathrm{d}\mu(y) \\ &= \int_V \varphi \circ g(y) \, \mathrm{mod}_K(g^*\omega).\end{aligned}$$

Now, in general, if M is an arbitrary K-analytic manifold of dimension d and ω a top-degree differential form on it, let us consider a *finite* family of charts (U_i, f_i) covering the support of $\varphi : M \to K$ together with a partition of unity

$$\sum_i \lambda_i \equiv 1$$

where the support of λ_i is contained in U_i. Then we set

$$\int_M \varphi \, \mathrm{mod}_K(\omega) = \sum_i \int_{f_i(U_i)} (\lambda_i \circ f_i^{-1}) \cdot (\varphi \circ f_i^{-1}) \cdot \mathrm{mod}_K((f_i^{-1})^*\omega).$$

Using the change of variable formula again, one checks that the right hand side (which is a finite sum) does not depend on the choices of charts and partitions of unity. □

2.4. Analytification of smooth schemes. Given a K-scheme of finite type, there exists a canonical way to endow its set of K-points with a topology satisfying two natural conditions.

Definition 2.8. Let X be a K-scheme of finite type. The *analytic topology* on $X(K)$ is the coarsest topology satisfying the following properties:

- for any Zariski-open subset $U \subset X$, its set $U(K)$ of K-points is open in $X(K)$;
- for every Zariski-open subset $U \subset X$ and any regular function $\varphi \in \mathscr{O}_X(U)$ the map $U(K) \to K$ induced by φ is continuous.

Defining a structure of a K-analytic manifold on a smooth K-scheme of finite type boils down to defining a subsheaf of the sheaf of continuous functions with values in K.

Definition 2.9. Let X be a *smooth* K-scheme of finite type.

Let U be an open subset of $X(K)$. We say that a function

$$f : U \longrightarrow K$$

is analytic at a point $x \in U$ if there exist a Zariski-open neighborhood $V \ni x$ in X, an immersion of K-schemes

$$i : V \hookrightarrow \mathbf{A}^n_K,$$

an open neighborhood $W \ni i(x)$ in $\mathbf{A}^n_K(K) = K^n$ together with an analytic function

$$g : W \longrightarrow K$$

such that

$$f = g \circ i$$

on an analytic neighborhood of $x \in X(K)$.

Proposition 2.2. *Via the previous definitions, the following holds.*

- *Any morphisms of smooth K-schemes induces a morphism of K-analytic manifolds; in particular,*
  - *open immersions induce open immersions of K-analytic manifolds;*
  - *closed immersions induce closed immersions of K-analytic manifolds.*
- *The structure of a K-analytic manifold on $\mathbf{A}^n_K(K) = K^n$ is the natural one: it coincides with the one from the previous sections.*
- *Any étale morphism of smooth K-schemes induces an étale morphisms of K-analytic manifolds (local isomorphisms).*

Proof. The first and second point are easy and left as an exercise.
The third point in an application of the local inversion Theorem 2.2. $\square$

The following proposition says that in the local non-Archimedean setting, rational points of closed subschemes are negligible.

Proposition 2.3. *Let K be a non-Archimedean local field and X a smooth K-scheme. Suppose that X is endowed with a measure μ associated to a gauge form, thanks to Proposition 2.1.*

Let

$$Z \subset X$$

be a closed subscheme of codimension at least 1. Then

$$\mu(Z(K)) = 0.$$

Proof. The question is local and it is sufficient to prove the following statement: if M is a submanifold of an open subscheme of K^d, of codimension $c \geqslant 1$ everywhere, then M has measure zero in K^d. Indeed, the (analytification of the) smooth locus of Z is locally of this form in the analytic topology. Then the singular locus of Z has codimension at least $c+1$ and we can conclude by induction on dimension.

So we assume that Z is smooth. Using the implicit function Theorem 2.1 and the change of variable formula Theorem 2.3 we reduce to the case

$$M = \{\underbrace{0, \ldots, 0}_{c \text{ times}}\} \times E^{d-c}(0,1)$$

inside $E^d(0,1)$, in particular observe that every polydisk

$$E^d(a,r) = \{x \in K^d \mid |x-a| \leqslant r\}$$

is isomorphic to the unit polydisk $E^d(0,1)$. Then the claim follows from Definition 2.5 page 13 (product measure) and the fact that $\{0\}$ has measure zero in K (more generally, any singleton is contained in a ball of arbitrary small radius, hence has measure zero). □

3. Batyrev's theorem

In this section, we apply techniques of p-adic integration to prove a theorem of Batyrev on the Betti numbers of Calabi-Yau varieties.

Definition 3.1. We call Calabi-Yau variety a *smooth* and *proper* complex variety with *trivial canonical bundle.*[a]

[a]In these notes we do not need to assume that Calabi-Yau are simply connected, as it is usually done in the literature.

Theorem 3.1 ([**Bat99**]). *Let X and Y be two Calabi-Yau varieties, which are birational. Then X and Y have the same Betti numbers.*

The proof of Theorem 3.1 relies on the Weil conjectures (proved by Dwork, Grothendieck and Deligne) to compute Betti numbers. We explain below the statements which are relevant to our proof, without dwelling too much upon the sophisticated tools of ℓ-adic cohomology that are used in Deligne's works.

The proof consists of the following steps.

(1) By a classical procedure called spreading-out in algebraic geometry, we can see a smooth proper complex algebraic variety as a smooth proper scheme over a finitely generated $\mathbf{Z}$-algebra A.
(2) There is a nice way to embed A into a valuation ring R, so that we can now see our varieties above R.
(3) We show that the two associated analytic varieties have equal p-adic volumes.
(4) It follows that both varieties have the same number of points over finite field. Using the Weil conjectures, we deduce equality of Betti numbers.

3.1. Hasse-Weil zeta functions and Betti numbers. Let X be a smooth and proper variety over $\mathbf{C}$. Since X is of finite type, it can be obtained by base change from a scheme $\mathscr{X}$ over a finitely generated integral $\mathbf{Z}$-algebra A, whose fraction field is contained in $\mathbf{C}$. We call $\mathscr{X}$ a spreading-out of X. By [**Gro66**, Thm. 8.10.5, Thm. 12.2.4.], up to localising A, we can assume that the structure morphism $\mathscr{X} \to \mathrm{Spec}(A)$ is proper and smooth as well.

DEFINITION 3.2. Let X be a finite type scheme over $\mathbf{F}_q$. The Hasse-Weil zeta function of X is the following power series $Z(X,T) \in \mathbf{Q}[\![T]\!]$:

$$Z(X,T) := \exp\left(\sum_{i\geq 1} \frac{\sharp X(\mathbf{F}_{q^i})}{i} T^i\right).$$

THEOREM 3.2 (Weil conjectures, [**Del74**, **Del80**]). *Let X be a smooth proper variety over $\mathbf{F}_q$. Then $Z(X,T)$ is a rational fraction in T of the form:*

$$Z(X,T) = \frac{\prod_i \det(1 - \mathrm{Fr}^* T | \mathrm{H}_c^{2i+1}(X, \mathbf{Q}_\ell))}{\prod_i \det(1 - \mathrm{Fr}^* T | \mathrm{H}_c^{2i}(X, \mathbf{Q}_\ell))}.$$

The characteristic polynomial $\det(1 - \mathrm{Fr}^* T | \mathrm{H}_c^i(X, \mathbf{Q}_\ell))$ *has integral coefficients, which are independent of the prime ℓ. Its complex roots have absolute value $q^{i/2}$.*

Note that the maximal ideals of $\mathrm{Spec}(A)$ have finite residue fields. This follows from the Nullstellensatz for Jacobson rings (see [**Eis95**, Thm. 4.19]). Alternatively, if A has no finite residue fields, then it contains the inverses of all primes. In that case, the Artin-Tate lemma [**Eis95**, Ex. 4.32] yields that $\mathbf{Q}$ of finite type over $\mathbf{Z}$, a contradiction. The existence of finite residue fields now allows to count points of $\mathscr{X}, \mathscr{Y}$ over finite fields.

PROPOSITION 3.1. *Let A be a finitely generated $\mathbf{Z}$-algebra, whose fraction field is contained in $\mathbf{C}$. Let $\mathscr{X}$ and $\mathscr{Y}$ be two smooth, proper A-schemes.*

Suppose that there exists a closed point $s \in \mathrm{Spec}(A)$ such that:

$$Z(\mathscr{X}_s, T) = Z(\mathscr{Y}_s, T).$$

Then the complex manifolds $X = \mathscr{X}(\mathbf{C})$ and $Y = \mathscr{Y}(\mathbf{C})$ have the same Betti numbers.

PROOF. Let ℓ be a prime number. We only consider geometric points of $\mathrm{Spec}(A)$ whose characteristic is different from ℓ. In particular, $\ell \neq \mathrm{char}(\kappa(s))$. By the Weil conjectures, $Z(\mathscr{X}_s, T) = Z(\mathscr{Y}_s, T)$ implies that the Betti numbers of $\mathscr{X}_s$ and $\mathscr{Y}_s$ in ℓ-adic cohomology are equal.

Since the structure morphism $f : \mathscr{X} \to \mathrm{Spec}(A)$ is smooth, we have that, for all $q \geq 0$, the cohomology groups $\mathrm{H}^q(\mathscr{X}_a, \mathbf{Q}_\ell)$, where a varies among geometric points of $\mathrm{Spec}(A)$, are isomorphic (this owes to the fact that the ℓ-adic constructible sheaves $R^q f_* \mathbf{Q}_\ell$ is locally constant, see [**DA73**, Exp. XVI, §2.]). The same holds for $\mathscr{Y}$.

Therefore, the Betti numbers of $\mathscr{X}_{\mathbf{C}}$ and $\mathscr{Y}_{\mathbf{C}}$ in ℓ-adic cohomology are equal. The result now follows from the comparison theorem between étale and singular cohomology [**DA73**, Exp. XI, Thm. 4.4.]. □

3.2. Weil's canonical measure and counts of points over finite fields. Batyrev's strategy to prove Theorem 3.1 is to count points of Calabi-Yau varieties over finite fields using a p-adic integral. This is done using a measure introduced by Weil on the analytification of the varieties at hand. We explain this below.

From now on, we fix a complete discrete valuation ring R with maximal ideal $\mathfrak{m}$, finite residue field k and fraction field K. Let q be the cardinal of k.

We saw earlier that a smooth K-scheme X gives rise to a non-Archimedean analytic manifold with underlying set $X(K)$, see Definition 2.8 page 16. Moreover, any (algebraic) differential form of top degree on X yields a differential form of top degree on $X(K)$ and therefore a measure on $X(K)$ by Proposition 2.1 page 15.

However, this measure depends on the differential form we chose and there may not be any non-zero differential form of top degree defined globally on X. When X is obtained by base change from a smooth R-scheme $\mathscr{X}$, Weil showed how to build a canonical measure on $\mathscr{X}(R)$ from *local* gauge forms, which is ultimately independent of all choices [**Wei82**, Ch.II, §2.].

CONSTRUCTION 3.1. Let $\mathscr{X}$ be a *smooth* R-scheme of pure relative dimension d. We will write $X = \mathscr{X}_K$ for short. By assumption, the sheaf of relative Kähler differentials

$$\Omega^d_{\mathscr{X}/R}$$

is locally invertible. Taking a trivialising affine open cover $(U_i)_{i\in I}$ of $\mathscr{X}$, we obtain gauge forms

$$\omega_i \in \Gamma\left(U_i, \Omega^d_{\mathscr{X}/R}\right) \quad i \in I$$

and associated measures

$$\mu_{\omega_i} \quad i \in I.$$

PROPOSITION 3.2. *The measures μ_{ω_i} glue to a measure on $\mathscr{X}(R)$, which does not depend on the choice of (U_i) and (ω_i).*

PROOF. This is essentially a consequence of the fact that the gauge forms we use are defined over R. Then the functions $f_{ij} = \frac{\omega_i}{\omega_j}$ have norm 1.

Note that

$$\mathscr{X}(R) = \bigcup_i U_i(R).$$

We first need to show that, for all measurable subsets $A \subseteq (U_i \cap U_j)(R)$, we have $\mu_{\omega_i}(A) = \mu_{\omega_j}(A)$. Up to taking an open cover, we may assume that A is contained in an analytic chart of $\mathscr{X}(R)$, with local coordinates $x_1, \ldots, x_d$. Let us write in coordinates: $\omega_i = f_i \cdot \mathrm{d}x$ and $\omega_j = f_j \cdot \mathrm{d}x$. Then, since $f_{ij}|_A = \frac{f_i}{f_j}$ takes values in $R^\times$ on $(U_i \cap U_j)(R)$, we obtain:

$$\mu_{\omega_i}(A) = \int_A |f_i(x)|\mathrm{d}x = \int_A |f_j(x)|\mathrm{d}x = \mu_{\omega_j}(A).$$

Likewise, we must show that $\mu_{\mathscr{X}}$ does not depend on the choice of (U_i) and (ω_i). This follows from the same computation, as the measures built from two choices (U_i', ω_i'), (U_i'', ω_i'') can be compared as above on the refined cover $(U_i' \cap U_j'')$. $\square$

DEFINITION 3.3 (Canonical measure). Let

$$\mu_{\mathscr{X}}$$

be the measure constructed above. It is often referred to as *Weil's canonical measure.*

THEOREM 3.3. [**Wei82**, Thm. 2.2.5.] *Let $\mathscr{X}$ be a smooth R-scheme of pure relative dimension d and let $\mu_{\mathscr{X}}$ be the associated canonical measure on $\mathscr{X}(R)$. Then:*

$$\mu_{\mathscr{X}}(\mathscr{X}(R)) = \frac{\sharp \mathscr{X}(k)}{q^d}.$$

PROOF. We can decompose $\mathscr{X}(R)$ as follows:

$$\mathscr{X}(R) = \bigsqcup_{\overline{x} \in \mathscr{X}(k)} B(\overline{x}) := \bigsqcup_{\overline{x} \in \mathscr{X}(k)} \left\{ x \in \mathscr{X}(R) \mid x|_{\mathrm{Spec}(k)} = \overline{x} \right\}.$$

So it is sufficient to show that, for all $\overline{x} \in \mathscr{X}(k)$:

$$\mu_{\mathscr{X}}\left(B(\overline{x})\right) = \frac{1}{q^d}.$$

Set $\overline{x} \in \mathscr{X}(k)$ and consider an affine chart of $\mathscr{X}$ around $\overline{x}$, of the form:

$$U = \operatorname{Spec}\left(R[T_1, \dots, T_n]/(f_1, \dots, f_r)\right),$$

such that (note that $n = r + d$):

$$\det\left(\frac{\partial f_i}{\partial T_j}(\overline{x})\right)_{\substack{1 \le i \le r \\ d+1 \le j \le n}} \neq 0.$$

Then by Hensel's Lemma 1.2 (giving a bijection) and the implicit function Theorem 2.1 (providing analyticity), the functions $T_1, \dots, T_d$ are local analytic coordinates over $B(\overline{x})$ and induce an analytic isomorphism

$$B(\overline{x}) \simeq \mathfrak{m}^d + (\overline{x_1}, \dots, \overline{x_d}).$$

Thus the volume of $B(\overline{x})$ under $\mu_{\mathscr{X}}$ can be computed as:

$$\mu_{\mathscr{X}}\left(B(\overline{x})\right) = \mu_{K^d}(\mathfrak{m}^d) = \frac{1}{q^d}.$$

This finishes the proof. □

3.3. p-adic volumes of birational K-trivial varieties. In this section, we finish the proof of Theorem 3.1. We first explain how to obtain p-adic Calabi-Yau manifolds from their complex analogues. Then we give the decisive argument: **birational p-adic Calabi-Yau manifolds have the same p-adic volume under Weil's canonical measure.** This owes to the fact that, when we compute p-adic volumes, we can neglect the locus where the two Calabi-Yau varieties are not isomorphic.

Let X and Y be two birational Calabi-Yau varieties. By spreading out, we can build from two smooth, proper, birational A-schemes with trivial canonical bundle, where A is an integral, finitely generated algebra with fraction field contained in $\mathbf{C}$ [**Gro66**, Thm. 8.3.11, Prop. 8.4.2, Thm. 8.5.2]. In order to recover counts of points over $\kappa(s)$ (s a closed point of $\operatorname{Spec}(A)$), we wish to use p-adic techniques as in the previous section. For this, we need to find a suitable base change from A to a ring of p-adic integers.

PROPOSITION 3.3. *Let A be an integral, finitely generated algebra with fraction field contained in $\mathbf{C}$. Then there exists a non-zero element $a \in A$ such that: for any maximal ideal*

$$\mathfrak{n} \subset A$$

not containing a, there exists an injective ring homomorphism

$$\varphi : A \to R$$

such that

$$\varphi^{-1}(\mathfrak{m}) = \mathfrak{n},$$

where R is the ring of integers of the unramified extension of $\mathbf{Q}_p$ with residue field $A/\mathfrak{n}$ (of characteristic p) and $\mathfrak{m} \subset R$ is its maximal ideal.

PROOF. We embed a generating set of A into R using Noether normalisation and Hensel's lemma as follows.

By Noether normalisation, there exist algebraically independent elements

$$T_1, \dots, T_d \in A$$

such that $A \otimes_{\mathbf{Z}} \mathbf{Q}$ is finite over $\mathbf{Q}[T_1, \dots, T_d] = \mathbf{Q}[\boldsymbol{T}]$. By generic smoothness [Har97, Ch. III, Cor. 10.7.], there exists

$$a \in \mathbf{Z}[\boldsymbol{T}] \subseteq A$$

such that the ring homomorphism

$$\mathbf{Z}[\boldsymbol{T}]\left[\frac{1}{a}\right] \to A\left[\frac{1}{a}\right]$$

is finite and étale.

Now, fix a maximal ideal $\mathfrak{n} \subset A$ not containing a. Let us first build a ring homomorphism

$$\varphi_0 : \mathbf{Z}[\boldsymbol{T}]\left[\frac{1}{a}\right] \to R$$

such that

$$\varphi_0^{-1}(\mathfrak{m}) = \frac{1}{a} \cdot (\mathbf{Z}[\boldsymbol{T}] \cap \mathfrak{n}).$$

A p-adic variant of Cantor's diagonal argument shows that R is uncountable, so $\mathrm{Frac}(R)$ has infinite transcendence degree over $\mathbf{Q}$ and we can find algebraically independent units

$$t_1, \dots, t_d \in R^\times.$$

Moreover, using Hensel's Lemma 1.2, we can find algebraic integers $r_1, \dots, r_d \in R$ such that

$$t_i r_i \bmod \mathfrak{m} = T_i \bmod \mathfrak{n}.$$

Since $a \notin \mathfrak{n}$, the injective ring homomorphism $\mathbf{Z}[\boldsymbol{T}] \to R$, $T_i \mapsto t_i r_i$ extends to

$$\varphi_0 : \mathbf{Z}[\boldsymbol{T}]\left[\frac{1}{a}\right] \hookrightarrow R,$$

with $\varphi_0^{-1}(\mathfrak{m}) = \mathbf{Z}[\boldsymbol{T}] \cap \mathfrak{n}$.

Finally, since $\mathbf{Z}[\boldsymbol{T}]\left[\frac{1}{a}\right] \to A\left[\frac{1}{a}\right]$ is finite and étale, we have a presentation:

$$A\left[\frac{1}{a}\right] = \mathbf{Z}[\boldsymbol{T}]\left[\frac{1}{a}\right][u_1, \dots, u_r]/(f_1, \dots, f_r),$$

where

$$\det\left(\frac{\partial f_i}{\partial u_j}\right) \in A\left[\frac{1}{a}\right]^\times.$$

Using Hensel's Lemma 1.2 again, we can extend φ_0 to $\varphi : A \to R$, by providing lifts of $u_i \bmod \mathfrak{n}$ to R. By construction, $\varphi^{-1}(\mathfrak{m}) = \mathfrak{n}$. Moreover, $\mathrm{Frac}(A)$ is an algebraic extension of $\mathbf{Q}(\boldsymbol{T})$. Since the latter injects into $\mathrm{Frac}(R)$ via φ_0, we get that φ is injective and we are done. □

REMARK 3.1. We will not make use of the fact (of independent interest, see [Cas86, Ch. 5]) that the ring homomorphism $A \to R$ is injective in the proof of Batyrev's theorem. Only the matching of residue fields is relevant to us here.

Applying base change, we may now assume that $\mathscr{X}$ and $\mathscr{Y}$ are two R-schemes satisfying our previous assumptions over A. Batyrev's Theorem 3.1 now follows from the following equality of p-adic volumes.

Proposition 3.4. *Let $\mathscr{X}$ and $\mathscr{Y}$ be two smooth, proper R-schemes of pure dimension d, with trivial canonical bundle.*

Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are birational over R. Then

$$\mu_{\mathscr{X}}(\mathscr{X}(R)) = \mu_{\mathscr{Y}}(\mathscr{Y}(R)).$$

Proof. We apply the change of variables formula. By assumption, there exist open R-subschemes $\mathscr{U} \subseteq \mathscr{X}$ and $\mathscr{V} \subseteq \mathscr{Y}$, together with an isomorphism $\phi : \mathscr{U} \xrightarrow{\sim} \mathscr{V}$. Consider a gauge form $\omega_{\mathscr{U}}$ (resp. $\omega_{\mathscr{V}}$) on $\mathscr{U}$ (resp. $\mathscr{V}$) (recall that $\mathscr{X}$ and $\mathscr{Y}$ have trivial canonical bundle). Then $\phi^*\omega_{\mathscr{V}}$ is a gauge form on $\mathscr{U}$, hence there exists $h \in \Gamma(\mathscr{U}, \mathscr{O}_{\mathscr{U}}^{\times})$ such that $\phi^*\omega_{\mathscr{V}} = h \cdot \omega_{\mathscr{U}}$. By change of variables, we obtain:

$$\int_{\mathscr{X}(R)\cap\mathscr{U}(K)} |h(x)| \mathrm{d}\mu_{\omega_{\mathscr{U}}}(x) = \int_{\mathscr{Y}(R)\cap\mathscr{V}(K)} \mathrm{d}\mu_{\omega_{\mathscr{V}}}.$$

We already know that $h(x) \in R^{\times}$ for all $x \in \mathscr{U}(R)$. By Lemma 3.1 below (and spreading out), we may assume that $\operatorname{codim}(\mathscr{X} \setminus \mathscr{U}) \geq 2$ (resp. $\operatorname{codim}(\mathscr{Y} \setminus \mathscr{V}) \geq 2$). Thus by Hartog's theorem [**Har97**, Ch. II, Prop. 6.3A.], h extends to a function $\tilde{h} \in \Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}}^{\times})$. So we further have that $h(x) \in R^{\times}$ for all $x \in \mathscr{X}(R) \cap \mathscr{U}(K)$.

Finally, since $\mathscr{X}(R) \cap \mathscr{U}(K)$ and $\mathscr{X}(R)$ (resp. $\mathscr{Y}(R) \cap \mathscr{V}(K)$ and $\mathscr{Y}(R)$) differ by the analytification of a subscheme of codimension at least 1, we obtain:

$$\mu_{\mathscr{X}}(\mathscr{X}(R)) = \mu_{\mathscr{Y}}(\mathscr{Y}(R)).$$

□

Lemma 3.1. *Let X, Y be two proper, smooth, complex d-dimensional varieties with trivial canonical bundle. Suppose that there exists a birational map $\phi : X \dashrightarrow Y$. Then ϕ induces an isomorphism between open subsets $U \subseteq X$ and $V \subseteq Y$ of codimension at least 2.*

Proof. This follows from Zariski's main theorem [**Har97**, Ch. III, Cor. 11.4]. Indeed, if a proper birational morphism of integral schemes with normal target $\psi : W_1 \to W_2$ is not an isomorphism above a locus $Z \subset W_2$, then $\psi^{-1}(Z)$ has dimension at most $\dim W_1 - 1 \geq \dim Z + 1$, hence $\operatorname{codim}_{W_2} Z \geq 2$. Applying this reasoning to a resolution of the closure of the graph of ϕ in $X \times Y$ gives the claim. □

Remark 3.2. Note that the change of variables done in the above proof only involves global gauge forms on $\mathscr{X}$ and $\mathscr{Y}$. It may seem unclear why we had to resort to local gauge forms to build the canonical measure in that setting. Actually, we use the fact that the canonical measure does not depend from the chosen gauge form to prove that the p-adic volume coincides with a count of points over the residue field; so we cannot avoid considering local gauge forms in the proof of Batyrev's Theorem 3.1.

Moreover, Theorem 3.1 holds more generally for K-equivalent varieties (not necessarily Calabi-Yau). In that context, global gauge forms may be unavailable and local gauge forms are also needed to prove the change of variables formula.

4. Some other algebro-geometric results we used

Étale and smooth morphisms (definition with charts)
Spreading-out
Generic smoothness [**Har97**, Ch. III, Cor. 10.7.]
Hartog's theorem [**Har97**, Ch. II, Prop. 6.3A.]
Zariski's main theorem, quasi-finite version [**Gro66**, Thm. 8.12.6.]
Comparison theorem between étale and singular cohomology [**DA73**, Exp. XI, Thm. 4.4.]

CHAPTER 2

Denef's formula and resolutions of singularities

In this chapter, we study a zeta function attached to a set of multivariate polynomials $\{f_i\}_{1\leq i\leq t}$, which was introduced by Igusa. This zeta function is defined as a p-adic integral and carries interesting information on the singularities of the affine scheme cut out by $\{f_i\}_{1\leq i\leq t}$. We prove a formula by Denef for Igusa's local zeta function in terms of an embedded resolution of singularities of the subscheme $V(f_i,\ i\in I)\subset \mathbf{A}^n_{\mathbf{C}}$. In the last section, we explain the monodromy conjecture. This conjecture relates poles of the local zeta function of an hypersurface to the Milnor fibre of the hypersurface. The techniques used in this chapter rely on reduction modulo p, in the spirit of Weil's conjectures, except that we now work with singular varieties.

1. Embedded resolutions of singularities

In this section, we define embedded resolutions of singularities. Embedded resolutions are a crucial tool in proving Denef's formula, as they allow to simplify the equations $\{f_i\}_{i\in I}$. The existence of such resolutions for general schemes is known only in characteristic zero, so we also discuss how these resolutions reduce modulo p.

1.1. Basic definitions. Over the course of this chapter, we will work over regular noetherian schemes. There are notions of local parameters on these schemes, which match local coordinates on the associated analytic manifolds. For the sake of conciseness, we only recall below the main facts that we will use in the proofs. We refer the interested reader to [**Eis95**] for the required commutative algebra foundations.

Definition 1.1. Let Y be a noetherian regular scheme and $y\in Y$. A system of parameters at y is a minimal set of generators for the maximal ideal $\mathfrak{m}_y\subset\mathcal{O}_{Y,y}$. Equivalently, these are elements of $\mathfrak{m}_y$ which form a basis of $\mathfrak{m}_y/\mathfrak{m}_y^2$.

Suppose that Y is a variety over a field F and $t_1,\dots,t_n$ are system of parameters at a closed point $y\in Y$. Then $t_1,\dots,t_n$ define a morphism $U\to\mathbf{A}^n_F$ defined on a neighbourhood $U\ni y$ which induces an isomorphism of tangent spaces, i.e. it is étale at y. This is the algebraic analogue of a local isomorphism in analytic geometry. We should then think of $t_1,\dots,t_n$ as local coordinates on Y at y.

Resolutions of singularities are typically constructed by blowing-up (see the discussion further below). Already for blowing up curve singularities, it is more convenient to embed a singular curve into a smooth space (for instance, the plane) and perform the blow-up on the ambient space. Besides, in order to compute Igusa's local zeta function, we have to require stronger properties for the resolution map: it must be an embedded resolution of singularities.

Fix an ambient regular scheme X and $Y\subset X$ a (singular) closed subscheme. An embedded resolution of singularities of the pair $Y\subset X$ is a birational map to X such that the inverse image of Y is a divisor with simple normal crossings. We briefly describe these divisors below.

Definition 1.2. Let Y be a regular noetherian scheme. A prime divisor on Y is a closed integral subscheme of Y of codimension 1. A (Weil) divisor is a formal linear combinations of prime divisors, with integer coefficients. A divisor $\sum_i N_i E_i$ is called effective if $N_i \geq 0$ for all i.

Effective divisors correspond bijectively to certain codimension 1 subschemes of Y. Consider an effective divisor $E = \sum_i N_i E_i$. Each E_i is cut out locally by a regular function f_i (this is a consequence of Krull's Hauptidealsatz - see [Eis95, Cor. 10.6.]). The subscheme associated to E is obtained by glueing the local subschemes defined by the equations $\prod_i f_i^{N_i}$. The corresponding reduced subscheme is the subscheme associated to $E_{\text{red}} := \sum_i E_i$.

Put more conceptually, the divisor E yields a line bundle on Y, called $\mathcal{O}_Y(E)$. The effective divisor E corresponds to a unique choice of section $s \in \Gamma(Y, \mathcal{O}_Y(E))$ (up to scaling) and the associated subscheme is the vanishing locus of s, whose defining ideal is $\mathcal{O}_Y(-E) \overset{s}{\hookrightarrow} \mathcal{O}_Y$.

Definition 1.3. An effective divisor E, with irreducible components E_i, $1 \leq i \leq t$, is said to have simple normal crossings if it is reduced and, at every point $y \in E$, there exists a system of parameters $t_1, \ldots, t_d \in \mathcal{O}_{Y,y}$ such that the defining ideal of E_i in $\mathcal{O}_{Y,y}$ is generated by t_i whenever $y \in E_i$.

We can now define embedded resolutions of singularities. Fix a base field $\mathbf{K}$. Consider a closed subscheme $X \subsetneq \mathbf{A}^m_{\mathbf{K}}$ defined by an ideal $I = (f_1, \ldots, f_r)$.

Definition 1.4. An embedded resolution of singularities of X is a projective morphism $h : Y \to \mathbf{A}^m_{\mathbf{K}}$ such that:

(1) Y is a smooth $\mathbf{K}$-variety;
(2) $h|_{h^{-1}(\mathbf{A}^m_{\mathbf{K}} \setminus X)} : h^{-1}(\mathbf{A}^m_{\mathbf{K}} \setminus X) \to \mathbf{A}^m_{\mathbf{K}} \setminus X$ is an isomorphism;
(3) $h^{-1}(X)$, seen as a reduced closed subscheme of Y, is a divisor E with simple normal crossings.

We call E_i, $1 \leq i \leq t$ the irreducible components of E. For $1 \leq i \leq t$, let N_i be the multiplicity of $h^{-1}(I) \cdot \mathcal{O}_Y$ along E_i and $\nu_i - 1$ the order of the form $h^*(\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m)$ along E_i. We call $(N_i, \nu_i)_{1 \leq i \leq t}$ the numerical data of the resolution h.

By a celebrated result of Hironaka [Hir64, Main Thm. II.], such an embedded resolution of singularities always exists when $\mathrm{char}(\mathbf{K}) = 0$. Embedded resolutions of singularities are generally built as successions of blow-ups. We will use singularities of plane curves as a running example. This only requires blowing up the affine plane in a point, which we explain below.

Example 1.1. The blowing-up of the affine plane at the origin is the following subscheme of $\mathbf{A}^2_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}}$:

$$\mathrm{Bl}_0(\mathbf{A}^2_{\mathbf{C}}) := \left\{ ((x,y), [u:v]) \in \mathbf{A}^2_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}} \mid yu = xv \right\}.$$

This scheme admits an open cover by two copies of the affine plane: the open $\{u \neq 0\}$ (resp. $\{v \neq 0\}$) is isomorphic to the affine plane with coordinates $x' = x$ and $y' = \frac{v}{u}$ (resp. $x'' = \frac{u}{v}$ and $y' = y$). The blowing-up map $\pi : \mathrm{Bl}_0(\mathbf{A}^2_{\mathbf{C}}) \to \mathbf{A}^2_{\mathbf{C}}$ is the projection on the first factor. Its expression in the local charts described above is:

$$\begin{array}{rcl} (x',y') & \mapsto & (x',x'y') \\ (x'',y'') & \mapsto & (x''y'',y''). \end{array}$$

A nice picture of the map π can be found in [**Har97**, Ch. I,§4.].

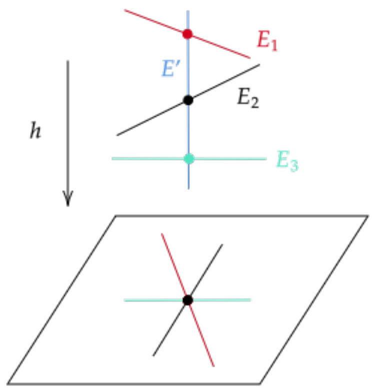
More generally, whenever a surface S contains an open subset isomorphic to an open subset U of the affine plane, one can construct the blow-up $\mathrm{Bl}_p(S)$ of a point $p \in U$ (say, mapped to the origin of the plane) by glueing $\pi^{-1}(U)$ and $S \setminus \{p\}$ along U using the aforementioned isomorphism. So we can iterate blow-ups. The definition of the blow-up of a scheme along a sheaf of ideals is of course much more flexible and general, but we will not use it in this lecture.

One can produce embedded resolutions of plane curves by blowing up their singular points in the ambient plane until we obtain a divisor with simple normal crossings. Here is an easy example: consider the plane curve $C \subset \mathbf{A}^2_{\mathbf{C}}$ cut out by the equation $xy(x+y) = 0$. This is not a divisor with simple normal crossing, as the three components intersect in the origin.

The inverse image of C in $\mathrm{Bl}_0(\mathbf{A}^2_{\mathbf{C}})$ is cut out by the following equations in the aforementioned charts:

$$\begin{array}{l} (x')^3 y'(1+y') = 0 \\ x''(y'')^3(1+x'') = 0. \end{array}$$

The inverse image of C has four irreducible components: E_1, E_2, E_3 cut out respectively by $y' = 0$, $x'' = 0$ and $1 + y' = \frac{v}{u}(1 + x'') = 0$ and E' cut out by $x' = \frac{u}{v} y'' = 0$. The corresponding divisor $E = E_1 + E_2 + E_3 + 3E'$ has simple normal crossings, since its components intersect transversally (two at a time), so we have obtained an embedded resolution of singularities. The numerical data of the resolution is, in order: $N_\bullet = (1, 1, 1, 3)$ and $\nu_\bullet = (1, 1, 1, 2)$ since, for instance, $\mathrm{d}x \wedge \mathrm{d}y = y'' \cdot \mathrm{d}x'' \wedge \mathrm{d}y''$.



EXAMPLE 1.2. Let us now compute an embedded resolution of the cusp $C = \{y^2 = x^3\} \subset \mathbf{A}^2_{\mathbf{C}}$.

The only singular point of the cusp is the origin, so we first blow up that point. In the affine charts discussed above, with coordinates:

$$\begin{array}{rcl} (x_1,y_1) & \mapsto & (x_1,x_1y_1) \\ (x_2,y_2) & \mapsto & (x_2y_2,y_2), \end{array}$$

the equation of the cusp becomes, respectively:

$$\begin{array}{l} x_1^2(y_1^2 - x_1) = 0 \\ y_2^2(1 - x_2^3 y_2) = 0. \end{array}$$

the reduced inverse image of C has no singular points in the second chart; in the first chart, though, we obtain a parabola touching a line tangentially in one point. So we need to further blow-up in the origin of the first chart.

Consider the blowing-up charts above the first chart above, with coordinates:

$$\begin{array}{rcl}(x_1', y_1') & \mapsto & (x_1', x_1' y_1') \\ (x_2', y_2') & \mapsto & (x_2' y_2', y_2').\end{array}$$

Then the equations of the blown-up curve become, respectively:

$$\begin{array}{l}(x_1')^3 (x_1' (y_1')^2 - 1) = 0 \\ (x_2')^2 (y_2')^3 (y_2' - x_2') = 0.\end{array}$$

Again, there are no singular points in the first chart and we obtain three lines intersecting in the origin of the second chart.

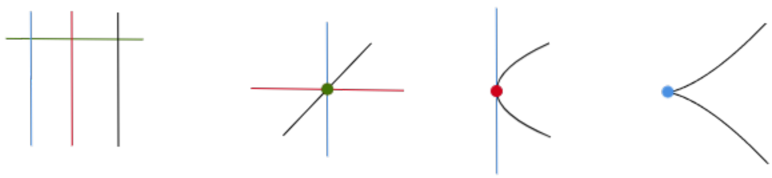
This can be resolved with a final blow-up in the origin of the second chart. We use the blowing-up charts with the following coordinates

$$\begin{array}{rcl}(x_1'', y_1'') & \mapsto & (x_1'', x_1'' y_1'') \\ (x_2'', y_2'') & \mapsto & (x_2'' y_2'', y_2'').\end{array}$$

and obtain the following equations for the blown-up curve:

$$\begin{array}{l}(x_1'')^6 (y_1'')^3 (y_1'' - 1) = 0 \\ (x_2'')^2 (y_2'')^6 (1 - x_2'') = 0.\end{array}$$

To sum up, the inverse image of C has four irreducible components: E_1 cut out (in the first chart) by $y_1'' - 1 = 0$ and E_1', E_2', E_3' cut out respectively by the equations $x_1'' = 0$, $y_1'' = 0$ and $x_2'' = 0$. The corresponding divisor is thus $E = E_1 + 6E_1' + 3E_2' + 2E_3'$. The numerical data of the resolution is, in order: $N_\bullet = (1, 6, 3, 2)$ and $\nu_\bullet = (1, 5, 3, 2)$.



Note that the divisor E_1 is isomorphic to C away from $(0,0) \in C$, so it is a resolution of singularities of C. This scheme is called the strict transform of C in the total space of the blow-up. The subscheme corresponding to $6E_1' + 3E_2' + 2E_3'$ is the fibre of the blow-up over $(0,0)$ and is called the exceptional divisor. The subscheme corresponding to E is called the total transform of C.

The strict transform of C after the first blow-up (the parabola in the first chart) is already smooth, so we already obtained a resolution of C after one blow-up. However, the additional blow-ups we performed above were necessary for the total transform to have simple normal crossings.

1.2. Good reduction modulo p. We now to discuss how to reduce an embedded resolution of singularities modulo p. Let us fix a number field $\mathbf{K}$. Consider an embedded resolution of singularities $h : Y \to \mathbf{A}^m_{\mathbf{K}}$ as above . Let $\mathfrak{p} \subseteq \mathcal{O}_{\mathbf{K}}$ be a non-zero prime ideal lying over $(p) \subseteq \mathbf{Z}$ with residue field $\mathcal{O}_{\mathbf{K}}/\mathfrak{p} \simeq k$. Let F be the completion of $\mathbf{K}$ with respect to the $\mathfrak{p}$-adic valuation and R its valuation ring. For p large enough, we may assume that X_F is defined over R, i.e. $f_i \in R[x_1, \ldots, x_m]$ for all $1 \leq i \leq r$. We further assume that not all f_i are equal to zero modulo $\mathfrak{p}$.

Since h is projective, we may consider Y_F as a closed subscheme of $\mathbf{P}^N_F \times_F X_F$. Consider Y_R (resp. E_R, $E_{i,R}$, $1 \leq i \leq t$) the scheme-theoretic closure of Y_F (resp. E_F, $E_{i,F}$, $1 \leq i \leq t$) in $\mathbf{P}^N_R \times_R X_R \supset \mathbf{P}^N_F \times_F X_F$.

Remark 1.1. The notion of scheme-theoretic closure is discussed in full generality in [Gro60, §9.5]. In our context, the scheme-theoretic closure of Y_F in $\mathbf{P}^N_R \times_R X_R$ is the topological closure of Y_F in $\mathbf{P}^N_R \times_R X_R$, endowed with the sheaf of rings $\mathcal{O}_{\mathbf{P}^N_R \times_R X_R}/\mathcal{I}$, where $\mathcal{I}$ is the kernel of $\mathcal{O}_{\mathbf{P}^N_R \times_R X_R} \to j_* \mathcal{O}_{Y_F}$ and j is the open immersion $\mathbf{P}^N_F \times_F X_F \hookrightarrow \mathbf{P}^N_R \times_R X_R$.

More explicitly, the scheme-theoretic closure can be computed locally along an affine open cover of $\mathbf{P}^N_R \times_R X_R$. So let us consider an ideal $J \subset F[x_1, \ldots, x_n]$, which corresponds to the closed subscheme $V(J) \subset \mathbf{A}^n_F$. Then the scheme-theoretic closure of $V(J)$ in $\mathbf{A}^n_R$ is $V(J \cap R[x_1, \ldots, x_n])$. Indeed, the topological closure of $V(J)$ is the vanishing locus of the maximal ideal $J' \subseteq R[x_1, \ldots, x_n]$ such that $F \cdot J' \subseteq J$. The ideal $J \cap R[x_1, \ldots, x_n]$ clearly satisfies this requirement and is the kernel of the morphism of rings $R[x_1, \ldots, x_n] \to F[x_1, \ldots, x_n]/J$.

Since Y_F is also defined over $A = \mathcal{O}_{\mathbf{K}}\left[\frac{1}{N}\right]$ for N large enough, one may wonder why we did not simply define Y_R as the base-change of Y_A along $A \to R$. It turns out that for p large enough, these constructions are equivalent (see for instance [Gro66, Cor. 9.4.5]).

Definition 1.5. We say that the resolution h has good reduction modulo $\mathfrak{p}$ if:
(1) Y_k is smooth;
(2) E_k is a divisor with simple normal crossings and $E_{i,k}$ is smooth for all $1 \leq i \leq t$;
(3) for $1 \leq i \neq j \leq t$, $E_{i,k}$ and $E_{j,k}$ have no common irreducible components.

Remark 1.2. The divisors $E_{i,k}$ may have several disjoint connected (or irreducible) components, e.g. $x(x - \varpi y) + a = 0$.

Example 1.3. Consider the plane curve $C = \{y^2 - px^2 - x^3 = 0\} \subset \mathbf{A}^2_{\mathbf{C}}$. Then the only singular point of C is the origin. After blowing up that point, we obtain the following equations in the usual blow-up charts:

$$\begin{array}{rcl} (x', y') & \mapsto & (x', x'y') \\ (x'', y'') & \mapsto & (x''y'', y''), \end{array}$$

$$\begin{array}{l} (x')^2((y')^2 - p - x') = 0 \\ (y'')^2(1 - (x'')^2(x''y'' + p)) = 0. \end{array}$$

The two components of the blown-up curve intersect transversally in two distinct points, so we obtain an embedded resolution of singularities. However, if we reduce modulo p, we obtain the cusp and the blown-up curve consists of a parabola meeting a line tangentially, as discussed in a previous example. So this embedded resolution has bad reduction modulo p.

In most cases however, an embedded resolution has good reduction. This is the content of the following:

Theorem 1.1 ([Den87, Thm. 2.4.]). *Let $X \subsetneq \mathbf{A}^m_{\mathbf{K}}$ be a closed subscheme and $h : Y \to \mathbf{A}^m_{\mathbf{K}}$ an embedded resolution of singularities of X. Then for almost all non-zero prime ideals $\mathfrak{p} \subseteq \mathcal{O}_{\mathbf{K}}$, the embedded resolution h has good reduction modulo $\mathfrak{p}$.*

Proof. This follows from general spreading-out results, which can be found in [Gro65, Cor. 6.8.7] and [Gro66, Cor. 9.5.2, Prop. 9.5.5]. □

We now collect geometric facts on the (good) reduction modulo $\mathfrak{p}$ of an embedded resolution. These will be useful in the proof of Denef's formula.

Proposition 1.1. [Den87, Prop. 2.6.] *Given an embedded resolution of singularities with good reduction modulo $\mathfrak{p}$ as above, we have that:*

1. *The scheme Y_R is flat over R, regular and integral.*
2. *Given $a \in Y_k$, and $\overline{g_1}, \dots, \overline{g_m}$ a system of parameters of $\mathcal{O}_{Y_k,a}$ (where $g_i \in \mathcal{O}_{Y_R,a}$, the elements $\varpi, g_1, \dots, g_m$ form a system of parameters of $\mathcal{O}_{Y_R,a}$.*
3. *The subscheme $Y_k \subset Y_R$ is integral, of codimension 1.*
4. *The morphism h_k is birational.*
5. *The subschemes $E_{i,R} \subset Y_R$ $(1 \leq i \leq t)$ are integral and of codimension 1.*

Proof of 1 and 2. We first check flatness. Since R is a discrete valuation ring, we just need to show that rings of regular functions on Y_R have no $\mathfrak{p}$-torsion. This can be checked on affine opens, so torsion-freeness follows from the fact mentioned above: for any $J \subseteq F[x_1, \dots, x_n]$, the scheme-theoretic closure of $V(J) \subseteq \mathbf{A}^n_F$ in $\mathbf{A}^n_R$ is $V(J \cap R[x_1, \dots, x_n])$. Moreover, since Y_F is integral, we have that Y_R is integral as well. Indeed this can also be checked on affine open subsets. Since the ring homomorphism

$$R[x_1, \dots, x_n]/(J \cap R[x_1, \dots, x_n]) \hookrightarrow K[x_1, \dots, x_n]/J$$

is injective, $R[x_1, \dots, x_n]/(J \cap R[x_1, \dots, x_n])$ is a domain whenever $K[x_1, \dots, x_n]/J$ is. This proves integrality.

We now check regularity. Let $a \in Y_k \subset Y_R$. Since $\mathcal{O}_{Y_k,a} \simeq \mathcal{O}_{Y_R,a}/\mathfrak{p}\mathcal{O}_{Y_R,a}$, the Krull dimension of $\mathcal{O}_{Y_R,a}$ is at most $\dim \mathcal{O}_{Y_k,a}+1$. On the other hand, since $\mathcal{O}_{Y_k,a}$ is regular, there exists a regular system of parameters $\overline{g_1}, \dots, \overline{g_m} \in \mathcal{O}_{Y_k,a}$, where $g_i \in \mathcal{O}_{Y_R,a}$. By Krull's principal ideal theorem [Eis95, Ch. 8, Thm. B], we obtain $\dim \mathcal{O}_{Y_R,a} \leq \dim \mathcal{O}_{Y_k,a} + 1$. Thus the maximal ideal of $\mathcal{O}_{Y_R,a}$ is generated by $\dim \mathcal{O}_{Y_R,a}$ generators, i.e. $\mathcal{O}_{Y_R,a}$ is regular and $\varpi, g_1, \dots, g_m$ form a regular system of parameters. This proves 2. On the other hand, if $a \in Y_F \subset Y_R$, then $\mathcal{O}_{Y_R,a}$ is regular, as Y_F is smooth over F. So all local rings of Y_R are regular, i.e. Y_R is regular. This finishes the proof of 1. □

Proof of 3. That $Y_k \subset Y_R$ has codimension 1 follows from 2. Let us show that Y_k is integral. Since h has good reduction modulo $\mathfrak{p}$, we have that Y_k is smooth, so it is enough to show that Y_k is connected. Indeed, irreducible components of Y_k must be disjoint, as local rings of Y_k are integral [Eis95, Cor. 10.14]. Thus the connected components of Y_k are its irreducible components and Y_k is irreducible if, and only if, it is connected.

We deduce that Y_k is connected from the fact that h_k is closed, surjective and has connected fibres. If $Y_k = Y_k' \sqcup Y_k''$ and Y_k', Y_k'' are both open (and closed), then fibres of h are contained either in Y_k' or in Y_k'', since we assumed that fibres are connected. Then $\mathbf{A}^m_k = h(Y_k') \sqcup h(Y_k'')$ and since $\mathbf{A}^m_k$ is connected, we obtain that either Y_k' or Y_k'' is empty, which shows that Y_k is connected.

It remains to show that h_k is closed, surjective and has connected fibres. Closedness follows from the fact that h_k is projective. Moreover, since h_R is birational and closed, its image in $\mathbf{A}^R_k$ must be dense and closed, so h_k is surjective. Finally, connectedness of fibres follows from Zariski's main theorem applied to h_R [Har97, Cor. 11.4]. □

Proof of 4. Since Y_k and $\mathbf{A}_k^m$ are integral, it suffices to show that h_k induces an isomorphism of function fields $k(\mathbf{A}_k^m) \simeq k(Y_k)$. Let η, θ be the generic points, respectively of Y_k and $\mathbf{A}_k^m$. We show the stronger fact that h_R induces an isomorphism of local rings $\mathcal{O}_{\mathbf{A}_R^m,\theta} \simeq \mathcal{O}_{Y_R,\eta}$.

By 3, Y_k is an integral subscheme of Y_R, of codimension 1. The same holds for $\mathbf{A}_k^m \subset \mathbf{A}_R^m$. Therefore, the local rings $\mathcal{O}_{\mathbf{A}_R^m,\theta}$ and $\mathcal{O}_{Y_R,\eta}$ are discrete valuation rings and h_R induces a morphism of local rings $\mathcal{O}_{\mathbf{A}_R^m,\theta} \to \mathcal{O}_{Y_R,\eta}$. Since h_R is birational, this homomorphism induces an isomorphism of fraction fields, so we actually have an isomorphism $\mathcal{O}_{\mathbf{A}_R^m,\theta} \simeq \mathcal{O}_{Y_R,\eta}$. Taking residue fields yields the isomorphism of function fields we wanted. $\square$

Proof of 5. Since $E_{i,R} \subsetneq Y_R$, we have that $\dim E_{i,R} < \dim Y_R$. On the other hand, $E_{i,R} \supset E_i$, so $\dim E_{i,R} \geq \dim E_i = \dim Y - 1 = \dim Y_R - 1$. This proves the claim on the codimension. The proof of integrality is the same as for Y_R. $\square$

2. Denef's formula

2.1. Igusa's local zeta function. We now introduce the local zeta function mentioned at the beginning of this chapter. We work over a non-archimedean local field F of characteristic zero. We call R its valuation ring and $k = \mathbf{F}_q$ its residue field.

Definition 2.1. Let $f = (f_1, \ldots, f_r)$, where $f_i \in R[x_1, \ldots, x_m]$. The local zeta function associated to f is the complex function defined by:

$$Z_f(s) := \int_{R^m} \|f(x)\|^s \mathrm{d}x, \ s \in \mathbf{C}.$$

Remark 2.1. Consider polynomials f_1,...,f_r as above and $x \in R^m$. Let

$$\mathrm{ev}_x : R[x_1, \ldots, x_m] \to R$$

be the evaluation morphism at x and $I = (f_1, \ldots, f_r) \subset R[x_1, \ldots, x_m]$. Then the norm $\|f(x)\|$ is given by $q^{-\mathrm{ord}_R(\mathrm{ev}_x(I))}$, where $\mathrm{ord}_R(\mathrm{ev}_x(I))$ is defined by

$$\mathrm{ev}_x(I) = (\varpi^{\mathrm{ord}_R(\mathrm{ev}_x(I))}) \subseteq R.$$

Therefore, the local zeta function Z_f only depends on the ideal $(f_1, \ldots, f_r)$, i.e. on the subscheme $V(f_1, \ldots, f_r) \subset \mathbf{A}_R^m$.

Example 2.1. Local zeta functions of monomials are particularly easy to compute. Set $f = x_1^{a_1} \ldots x_m^{a_m}$. Then by Fubini's theorem:

$$Z_f(s) = \prod_{i=1}^m \left(\int_R |x|^{a_i s} \mathrm{d}x \right) = \prod_{i=1}^m \left(\sum_{n \geqslant 0} (1 - q^{-1}) q^{-n(a_i s + 1)} \right) = \prod_{i=1}^m \frac{1 - q^{-1}}{1 - q^{-(a_i s + 1)}}.$$

Example 2.2. When $f_1, \ldots, f_r$ are no longer monomials, the computations become more cumbersome. Consider for instance $f(x, y) = xy(x + y)$.

By symmetry, we have:

$$Z_f(s) = 2 \int_{|x| > |y|} |xy(x+y)|^s \mathrm{d}x\mathrm{d}y + \int_{|x|=|y|} |xy(x+y)|^s \mathrm{d}x\mathrm{d}y.$$

The first term is readily computable:

$$\int_{|x|>|y|} |xy(x+y)|^s \mathrm{d}x\mathrm{d}y = \sum_{m\geq 0}\sum_{n>m}(1-q^{-1})^2 q^{-m-n}\cdot q^{-2ms-ns}$$
$$= q^{-(s+1)}\frac{1-q^{-1}}{1-q^{-(s+1)}}\frac{1-q^{-1}}{1-q^{-(3s+2)}}.$$

We compute the second term using the change of variables $y = ux$:

$$\begin{aligned}\int_{|x|=|y|} |xy(x+y)|^s \mathrm{d}x\mathrm{d}y &= \int_{R\times R^\times} |x^3(1+u)|^s |x| \mathrm{d}x\mathrm{d}u \\ &= \left(\int_R |x|^{3s+1}\mathrm{d}x\right)\left(\int_{R^\times} |1+u|^s \mathrm{d}u\right) \\ &= \frac{1-q^{-1}}{1-q^{-(3s+2)}}\cdot\left(\frac{q-2}{q} + q^{-(s+1)}\cdot\int_R |v|^s \mathrm{d}v\right) \\ &= \frac{1-q^{-1}}{1-q^{-(3s+2)}}\cdot\left(\frac{q-2}{q} + q^{-(s+1)}\frac{1-q^{-1}}{1-q^{-(s+1)}}\right),\end{aligned}$$

where we used the change of variables $u = -1 + \varpi v$. Summing up, we obtain:

$$Z_f(s) = \frac{(1-q^{-1})(1-2q^{-1})}{1-q^{-(3s+2)}} + 3q^{-(s+1)}\frac{1-q^{-1}}{1-q^{-(s+1)}}\frac{1-q^{-1}}{1-q^{-(3s+2)}}.$$

2.2. Proof of Denef's formula. We now prove a formula due to Denef, which computes the local zeta function of a polynomial mapping from an embedded resolution of singularities. The proof relies on change of variables and our previous computation of local zeta functions of monomials.

THEOREM 2.1 ([Den87, Thm. 3.1.],[VZG08, Thm. 2.10.]). *Let* $\mathbf{K}$ *be a number field. Let* $X = V(I) = V(f_1,\dots,f_r) \subseteq \mathbf{A}^m_{\mathbf{K}}$ *and* $h : Y \to \mathbf{A}^m_{\mathbf{K}}$ *be an embedded resolution of singularities of* $X \subset \mathbf{A}^m_{\mathbf{K}}$*, with numerical data* $(N_i,\nu_i)_{1\leq i\leq t}$*. Let* $\mathfrak{p} \subseteq \mathcal{O}_{\mathbf{K}}$ *be a non-zero prime ideal with residue field* k*. Suppose that* h *has good reduction modulo* $\mathfrak{p}$*. Then:*

$$Z_f(s) = q^{-m}\cdot\sum_{I\subseteq\{1,\dots,t\}} c_I\cdot\prod_{i\in I}\frac{(q-1)q^{-N_is-\nu_i}}{1-q^{-N_is-\nu_i}},$$

where

$$c_I = \sharp\{a\in Y(k) \mid a\in E_i(k) \Leftrightarrow i\in I\}$$

and $q = \sharp k$*.*

EXAMPLE 2.3. Let us apply Denef's formula to the previous example $f = xy(x+y)$. Recall the resolution obtained by blowing up the origin of the plane. The strict transform consists of three disjoint affine lines E_1, E_2, E_3. The exceptional divisor E' is isomorphic to $\mathbf{P}^1_R$ and meets each component of the strict transform in exactly one (k)-point. The numerical data of the resolution is $N_\bullet = (1,1,1,3)$ and $\nu_\bullet = (1,1,1,2)$.

Summing up all contributions, we obtain:

$$\begin{aligned} Z_f(s) &= q^{-2}(q-2)(q-1) \\ &+ 3q^{-2}(q-1)\frac{q^{-(s+1)}(q-1)}{1-q^{-(s+1)}} \\ &+ q^{-2}(q-2)\frac{q^{-(3s+2)}(q-1)}{1-q^{-(3s+2)}} \\ &+ 3q^{-2}\frac{q^{-(s+1)}(q-1)}{1-q^{-(s+1)}}\frac{q^{-(3s+2)}(q-1)}{1-q^{-(3s+2)}}, \end{aligned}$$

which adds up to:

$$Z_f(s) = \frac{(1-q^{-1})(1-2q^{-1})}{1-q^{-(3s+2)}} + 3q^{-(s+1)}\frac{1-q^{-1}}{1-q^{-(s+1)}}\frac{1-q^{-1}}{1-q^{-(3s+2)}}.$$

Thus we recover the result of our computation by hand.

We now turn to the proof of Denef's formula. The key step is the following:

Proposition 2.1. *Let $X = V(I) = V(f_1, \ldots, f_r) \subseteq \mathbf{A}^m_{\mathbf{K}}$ and $h : Y \to \mathbf{A}^m_{\mathbf{K}}$ be an embedded resolution of singularities of $X \subset \mathbf{A}^m_{\mathbf{K}}$, with numerical data $(N_i, \nu_i)_{1 \leq i \leq t}$. Let $\mathfrak{p} \subseteq \mathcal{O}_{\mathbf{K}}$ be a non-zero prime ideal with residue field k. Let us call R the completion of $\mathcal{O}_{\mathbf{K}}$ at $\mathfrak{p}$. Suppose that h has good reduction modulo $\mathfrak{p}$.*

Consider $a \in Y(k)$. Define

$$B_a = \{y \in Y(R) \mid \overline{y} = a\}$$

and

$$T_a = \{1 \leq i \leq t \mid a \in E_i(k)\}.$$

Then there is a bijective analytic mapping

$$\varphi : \mathfrak{p}R^m \to B_a$$

such that:

- *for all $z \in \mathfrak{p}R^m$, $\|f \circ h \circ \varphi(z)\| = \prod_{i \in T_a} |z_i|^{N_i}$;*
- *$(h \circ \varphi)^* \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m = \prod_{i \in T_a} z_i^{\nu_i - 1} \cdot dz_1 \wedge \ldots \wedge dz_m$.*

Proof. The following lemmas give a simple, explicit expression of $f \circ h$ and $h^*\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m$ in local analytic coordinates on B_a.

Lemma 2.1. *In the local ring $\mathcal{O}_{Y_R,a}$, the ideal generated by $h^{-1}(I)$ is generated by a function of the form:*

$$u \cdot \prod_{i \in T_a} g_i^{N_i},$$

where u is a unit in $\mathcal{O}_{Y_R,a}$ and g_i is a generator of the ideal of $E_{i,R}$ in $\mathcal{O}_{Y_R,a}$.

Lemma 2.2. *In the module of differentials $\Omega_{\mathcal{O}_{Y_R,a}/R}$, the form $h^*(\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m)$ has the following expression:*

$$h^*(\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m) = v \cdot \prod_{i \in T_a} g_i^{\nu_i - 1} dg_1 \wedge \ldots \wedge dg_m,$$

where v is a unit in $\mathcal{O}_{Y_R,a}$ and $g_1, \ldots, g_m$ are part of a regular system of parameters for $\mathcal{O}_{Y_R,a}$, which include g_i, $i \in T_a$ as in the previous lemma.

LEMMA 2.3. *Let $g_1, \ldots, g_m$ be a regular system of parameters for $\mathcal{O}_{Y_R,a}$ as in the previous lemma. Then the map:*

$$\begin{array}{rcl} B_a & \to & \mathfrak{p}R^m \\ x & \mapsto & (g_1(x), \ldots, g_m(x)) \end{array}$$

is a bijective analytic mapping.

Let $\varphi : \mathfrak{p}R^m \to B_a$ be the mapping inverse to $x \mapsto g(x)$. Using change of variables, first along h, then along φ, we obtain:

$$\begin{aligned} Z_f(s) &= \sum_{a \in Y(k)} \int_{B_a} \|f \circ h\|^s |h^*(\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m)| \\ &= \sum_{a \in Y(k)} \int_{\mathfrak{p}R^m} \prod_{i \in T_a} |x_i|^{N_i s + \nu_i - 1} \mathrm{d}x \\ &= \sum_{a \in Y(k)} q^{-m} \cdot \prod_{i \in T_a} \frac{(q-1)q^{-N_i s - \nu_i}}{1 - q^{-N_i s - \nu_i}}, \end{aligned}$$

which yields the desired formula. □

PROOF OF LEMMA 2.1. We analyse the equation at a of the divisor cut out by $h^{-1}(I)$ in Y_R. We see $a \in Y(k)$ as a closed point in Y_R. By Proposition 1.1, Y_R is regular, so the local ring $\mathcal{O}_{Y_R,a}$ is factorial [**Eis95**, Thm. 19.19.]. Note that, since h is an embedded resolution of singularities, the ideal generated by $h^{-1}(I)$ in $\mathcal{O}_{Y_R,a}$ is generated by a single element $\tilde{f}$.

Since by Proposition 1.1, the subscheme $E_{i,R} \subset Y_R$ $(1 \leq i \leq t)$ is a divisor, the corresponding ideal in $\mathcal{O}_{Y_R,a}$ is principal. Call $g_i \in \mathcal{O}_{Y_R,a}$ an irreducible generator of the ideal of $E_{i,R}$. As the multiplicity of $\tilde{f}$ along $E_{i,R}$ is evaluated at the generic point $\eta_i \in E_{i,R}$ and $\eta_i \in Y_F$, we obtain that:

$$\tilde{f} = u \cdot \prod_{i \in T_a} g_i^{N_i},$$

where $u \in \mathcal{O}_{Y_R,a}$ is relatively prime to g_i $(i \in T_a)$.

We conclude by proving that u is a unit in $\mathcal{O}_{Y_R,a}$. By contradiction, suppose that u is not a unit and consider an irreducible factor of u, called g. Since u is relatively prime to g_i (for all $i \in T_a$), the subscheme of Y_R cut out by g has codimension 1 and is contained in Y_k. By Proposition 1.1, the scheme Y_k is integral, also of codimension 1, so we obtain that g, hence $\tilde{f}$ vanishes on Y_k. This contradicts the fact that h_k is birational (Proposition 1.1) and not all f_i are equal to zero modulo $\mathfrak{p}$, so we are done. □

PROOF OF LEMMA 2.2. As in the proof of the previous Lemma, we analyse the differential form $h^*(\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m)$ locally at a. Since

$$\Omega_{\mathcal{O}_{Y_R,a}/R} \otimes_R k \simeq \Omega_{\mathcal{O}_{Y_k,a}/k},$$

we can find generators of the R-module $\Omega_{\mathcal{O}_{Y_R,a}/R}$ by lifting generators of the k-vector space $\Omega_{\mathcal{O}_{Y_k,a}/k}$ (easy instance of Nakayama's lemma).

Since, by assumption, Y_k is regular and E_k has simple normal crossings at a, with distinct components $E_{i,k}$ $(i \in T_a)$, the elements $\overline{g_i}$ $(i \in T_a)$ obtained from g_i by reduction modulo $\mathfrak{p}$ are part of a system of parameters of $\mathcal{O}_{Y_k,a}$. Let us call $\overline{g_1}, \ldots, \overline{g_m}$ such a system of parameters. Then the conormal sequence [**Eis95**, Prop. 16.3.] implies that $\Omega_{\mathcal{O}_{Y_k,a}/k}$

is generated by $d\overline{g_1}, \dots, d\overline{g_m}$. Therefore, $\Omega_{\mathcal{O}_{Y_k,a}/k}$ is generated by $dg_1, \dots, dg_m$. As in the proof of the previous lemma there exists an element $v \in \mathcal{O}_{Y_R,a}$ such that:

$$h^*(\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_m) = v \cdot \prod_{i \in T_a} g_i^{\nu_i - 1} \cdot dg_1 \wedge \dots \wedge dg_m.$$

Then a similar reasoning shows that v must be a unit in $\mathcal{O}_{Y_R,a}$. □

Proof of Lemma 2.3. Let us first note that for all $x \in B_a$, $g_i(x) \in \mathfrak{p}R$ for $1 \leq i \leq m$. Indeed, $\overline{g_i}$ vanishes at $a \in Y_k$ and $\overline{x} = a$ by definition. So the map $g : B_a \to \mathfrak{p}R^m$ is well-defined.

We now prove that g is bijective. Given $(z_1, \dots, z_m) \in \mathfrak{p}R^m$, the set of points $x \in B_a$ such that $g(x) = z$ is in bijection with homomorphisms of local R-algebras:

$$\mathcal{O}_{Y_R,a}/(g_1 - z_1, \dots, g_m - z_m) \to R,$$

where x corresponds to the evaluation morphism at x. Now by Proposition 1.1, $\varpi, g_1 - z_1, \dots, g_m - z_m$ is a system of parameters of the regular local ring $\mathcal{O}_{Y_R,a}$, so $\mathcal{O}_{Y_R,a}/(g_1 - z_1, \dots, g_m - z_m)$ is a regular local ring of dimension 1 [**Eis95**, Ch. 10], hence a discrete valuation ring [**Eis95**, §11.1.]. Since R is complete and $\mathcal{O}_{Y_R,a}/(g_1 - z_1, \dots, g_m - z_m)$ is unramified over R, with residue field k, we obtain that

$$\mathcal{O}_{Y_R,a}/(g_1 - z_1, \dots, g_m - z_m) \simeq R.$$

Therefore, there exists a unique solution to the equation $g(x) = z$ and we are done. □

3. The monodromy conjecture

In this section, we explain an important conjecture in singularity theory: Igusa's monodromy conjecture. The conjecture relates topological invariants of a singular hypersurface $\{f = 0\} \subset \mathbf{C}^n$ to the local zeta function $Z_f(s)$, for $f \in \mathbf{Q}[x_1, \dots, x_n]$. The conjecture remains wide open to date. The only dimension $n-1$ where it has been proven in full generality is $n = 2$, i.e. for singular plane curves. We refer [**Vey24**] to for a survey.

Before defining precisely the topological invariants involved in the conjecture, let us informally explain why they should be related. The local zeta function $Z_f(s)$ encodes the counts of points of the hypersurface $\{f = 0\}$ over $\mathbf{Z}/p^k\mathbf{Z}$, for all $k \geq 1$ (see Exercise 4). This information concerns the scheme $V(f) \subset \mathbf{A}^n_{\mathbf{Z}_p}$, defined over $\mathbf{Z}_p$. Now, the valuation ring $\mathbf{Z}_p$ should be thought of as an arithmetic version of $\mathbf{C}[\![t]\!]$, that is, as the coordinate ring of a formal disk around $0 \in \mathbf{C}$. From this point of view, it seems natural that we consider invariants of the map $f : \mathbf{A}^n_{\mathbf{C}} \to \mathbf{A}^1_{\mathbf{C}}$ over a small disk centered at the origin $0 \in \mathbf{C}$. Indeed, the monodromy conjecture is about topological invariants of so-called Milnor fibres.

Milnor fibres are built from the following fibration, which was first studied by Milnor (see [**Dim92**, Ch. 3] for details).

Theorem 3.1. *Let $f \in \mathbf{C}[x_1, \dots, x_n] \backslash \mathbf{C}$ satisfy $f(0) = 0$. Consider an open punctured disk $D^*_\delta \subset \mathbf{C} \setminus \{0\}$, centered at $0 \in \mathbf{C}$, of radius δ and an open ball $B_\epsilon \subset \mathbf{C}^n$, centered at $0 \in \mathbf{C}^n$, of radius ϵ. Then for $\epsilon >> \delta > 0$ small enough, the map:*

$$f : B_\epsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta$$

is a smooth, locally trivial fibration.

If we restrict f over a small circle S^1 around the origin $0 \in \mathbf{C}$, then all fibres are diffeomorphic. Let us call F the fibre at an arbitrary base point $p_0 \in S^1$. The fibration over S^1 is determined by its monodromy diffeomorphism $h : F \xrightarrow{\sim} F$. The diffeomorphism h is obtained by following local trivialisations of f along a one-turn loop on the base circle.

DEFINITION 3.1. The fibration $f^{-1}(S^1) \to S^1$ described above is called the Milnor fibration of f at $0 \in \mathbf{C}^n$. The fibre

$$F := f^{-1}(p_0)$$

is called the Milnor fibre of f at $0 \in \mathbf{C}^n$. The algebraic monodromy is the collection of linear automorphisms:

$$M_i := h^* : \mathrm{H}^i(F, \mathbf{C}) \to \mathrm{H}^i(F, \mathbf{C}),\ 0 \leq i \leq 2(n-1).$$

A monodromy eigenvalue of f at $0 \in \mathbf{C}^n$ is an eigenvalue of at least one of the operators M_i above. Given $a \in \{f = 0\}$, we define the Milnor fibration, Milnor fibre and the monodromy operators at a similarly, by considering the map $z \mapsto f(a + z)$.

We collect below some (non-trivial) facts on Milnor fibres and monodromy eigenvalues, which will be useful when checking the monodromy conjecture on examples.

PROPOSITION 3.1. *Let $f \in \mathbf{C}[x_1, \ldots, x_n] \setminus \mathbf{C}$ and $a \in \{f = 0\}$. Call F the Milnor fibre of f at a. Then the following hold:*

- *(Monodromy Theorem) The monodromy eigenvalues at a are all roots of unity.*
- *For $i \geq n$, we have $H^i(F, \mathbf{C}) = 0$.*
- *If a is a smooth point of $\{f = 0\}$, then $\mathrm{H}^\bullet(F, \mathbf{C}) = \mathrm{H}^0(F, \mathbf{C}) = \mathbf{C}$ and the algebraic monodromy is equal to 1.*
- *If a is an isolated singular point of $\{f = 0\}$, then:*

$$\mathrm{H}^\bullet(F, \mathbf{C}) = \mathrm{H}^0(F, \mathbf{C}) \oplus \mathrm{H}^{n-1}(F, \mathbf{C}) = \mathbf{C} \oplus \mathrm{H}^{n-1}(F, \mathbf{C})$$

and the algebraic monodromy on $\mathrm{H}^0(F, \mathbf{C})$ is equal to 1.

The monodromy conjecture is about monodromy eigenvalues of a polynomial $f \in \mathbf{Q}[x_1, \ldots, x_n] \setminus \mathbf{Q}$. Before we state the conjecture, note that, by Denef's formula, the local zeta function $Z_f(s)$ is a rational fraction in p^{-s} with coefficients in $\mathbf{C}$. Its poles are located among the complex numbers $s = -\frac{\nu_i}{N_i} + \frac{2\pi i k}{N_i}$, $k \in \mathbf{Z}$, where (N_i, ν_i) is the numerical data of some embedded resolution of $\{f = 0\} \subset \mathbf{A}^n_{\mathbf{C}}$. In particular, the real parts of poles are rational numbers.

Conjecture 3.1. *Let $f \in \mathbf{Q}[x_1, \ldots, x_n] \setminus \mathbf{Q}$. For almost all primes p, we have the following: if $s_0 \in \mathbf{C}$ gives a pole of $Z_f(s)$, then $e^{2\pi i \mathrm{Re}(s_0)}$ is a monodromy eigenvalue of f at some point of $\{f = 0\}$.*

REMARK 3.1. There is also a local version of the monodromy conjecture. Suppose that $f(0) = 0$, and call $B_k := p^k \mathbf{Z}_p^n$ Let us define:

$$Z_{f,B_k}(s) := \int_{B_k} |f(x)|^s \mathrm{d}x.$$

Then the following is also conjectured: for almost all primes p and for k large enough, if s_0 gives a pole of $Z_{f,B_k}(s)$, then $e^{2\pi i \mathrm{Re}(s_0)}$ is a monodromy eigenvalue of f at points in a neighbourhood of 0.

As for the local zeta function, some information on monodromy eigenvalues can be recovered from an embedded resolution of $\{f = 0\} \subset \mathbf{A}^n_{\mathbf{C}}$. This is the content of A'Campo's formula for the monodromy zeta function.

Definition 3.2. Let $f \in \mathbf{C}[x_1, \ldots, x_n] \setminus \mathbf{C}$ and $a \in \{f = 0\}$. Let F_a be the Milnor fibre of f at a and $P_i(t)$ the characteristic polynomial of algebraic monodromy on $H^i(F_a, \mathbf{C})$. The monodromy zeta function of f at a is the following rational fraction:

$$\zeta_{f,a}(t) := \prod_{i=0}^{n-1} P_i(t)^{(-1)^{i+1}} = \frac{P_1(t)P_3(t)\cdots}{P_0(t)P_2(t)\cdots}.$$

Theorem 3.2. [A'C75] *Let $f \in \mathbf{C}[x_1, \ldots, x_n] \setminus \mathbf{C}$ and $a \in \{f = 0\}$. Let $h : Y \to \mathbf{A}^n_{\mathbf{C}}$ be an embedded resolution of $\{f = 0\} \subset \mathbf{A}^n_{\mathbf{C}}$, with numerical data $(N_i, \nu_i)_{1 \le i \le t}$. For $1 \le i \le t$, let us call $E_i^\circ := E_i \setminus \bigcup_{j \ne i} E_j$. Then we have:*

$$\zeta_{f,a}(t) = \prod_{i=1}^{t} (t^{N_i} - 1)^{-\chi(E_i^\circ \cap h^{-1}(a))},$$

where χ denotes the Euler characteristic. In particular, when $a = 0$ is an isolated singular point, we have:

$$\begin{array}{ll} P_{n-1}(t) = (t-1) \cdot \prod_{i=1}^{t'} (t^{n_i} - 1)^{-\chi(E_j^\circ)} & n \text{ even}, \\ P_{n-1}(t) = \frac{1}{t-1} \cdot \prod_{i=1}^{t'} (t^{n_i} - 1)^{\chi(E_j^\circ)} & n \text{ odd}, \end{array}$$

where we assume that $h^{-1}(0)$ is the union of E_i, $1 \le i \le t'$.

With these tools at hand, one can check the monodromy conjecture on some plane curve singularities.

Example 3.1. Consider the equation of the cusp in the affine plane, i.e. $f = y^2 - x^3$. Recall that we computed an embedded resolution of singularities h, so that $f \circ h$ is the equation of a divisor $E = E_1 + 6E_1' + 3E_2' + 2E_3'$, with numerical data $N_\bullet = (1, 6, 3, 2)$ and $\nu_\bullet = (1, 5, 3, 2)$.

Note that $h^{-1}(0)$ is the union of E_1', E_2', E_3', which are all isomorphic to $\mathbf{P}^1_{\mathbf{C}}$. Moreover, E_1' meets the each of E_1, E_2', E_3' in exactly one point and E_1, E_2', E_3' do not intersect pairwise. Thus, $(E_1')^\circ$ is isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ minus three points, whereas $(E_2')^\circ$ and $(E_3')^\circ$ are isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ minus one point. Hence we get $\chi((E_1')^\circ) = -1$ and $\chi((E_2')^\circ) = \chi((E_3')^\circ) = 1$ and:

$$P_1(t) = (t-1)(t^2-1)^{-1}(t^3-1)^{-1}(t^6-1) = 1 - t + t^2.$$

On the other hand, Denef's formula gives us the local zeta function of the cusp:

$$Z_f(s) = 1 - \frac{(1-q^{-s})(q^{-1} + (q-1)q^{-(s+3)} + (q-1)q^{-(5s+6)} - q^{-(6s+6)})}{(1-q^{-(s+1)})(1-q^{-(6s+5)})}.$$

The poles of $Z_f(s)$ have real parts -1, $-5/6$ and $e^{-2\pi i} = 1$, $e^{-5\frac{2\pi i}{6}} = e^{i\frac{\pi}{3}}$ are roots of $P_0(t) = t - 1$, $P_1(t) = 1 - t + t^2$ respectively. Thus we have checked the monodromy conjecture for the cusp.

CHAPTER 3

Adelic spaces: integration, Fourier analysis, and applications

1. Welcome to the adelic world

Let K be a global field, that is to say either a number field or the function field of a curve above a finite field.

1.1. First definitions.

DEFINITION 1.1. The space of adèles of K is the restricted product

$$\begin{aligned}\mathbb{A}_K &= \left\{ (x_v) \in \prod_{v\in M_K} K_v \;\middle|\; x_v \in \mathcal{O}_v \text{ for all but finitely many } v \in M_K \right\} \\ &= \prod_{v\in M_K}{}' (K_v, \mathcal{O}_v) \\ &= \varinjlim_{\substack{S\subset M_K \\ \text{finite}}} \prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v.\end{aligned}$$

The following result tells us that many properties can be deduced from the study of the adèles of $\mathbf{Q}$ and $\mathbf{F}_q(t)$. Its proof is rather technical and will probably not be treated in these notes.

LEMMA 1.1. *Let L be a finite separable extension of a global field K. Then*

$$\mathbb{A}_K \otimes_K L \simeq \mathbb{A}_L$$

as locally compact abelian groups, sending $L \simeq K \otimes_K L$ isomorphicaly onto $L \hookrightarrow \mathbb{A}_L$.

Our global field K embeds diagonally into $\mathbb{A}_K$.

PROPOSITION 1.1. *The diagonal embedding*

$$K \hookrightarrow \mathbb{A}_K$$

identifies K with a discrete cocompact subgroup of $\mathbb{A}_K$.

PROOF. By the previous lemma, it is enough to treat the cases of $\mathbf{Q}$ and $\mathbf{F}_q(t)$. First, the open subset

$$U = \{|x_\infty| < 1 \text{ and } |x_p|_p \leqslant 1 \text{ for all primes } p\}$$

is an open subset of $\mathbb{A}_\mathbf{Q}$ such that $U \cap \mathbf{Q} = \{0\}$, showing that $\mathbf{Q}$ is discrete in $\mathbb{A}_\mathbf{Q}$. Then we construct a compact fundamental domain for $\mathbf{Q}$, by taking

$$W = \left\{ |x_\infty| \leqslant \frac{1}{2} \text{ and } |x_p|_p \leqslant 1 \text{ for all primes } p \right\}.$$

For a general number field, one can take

$$D = \prod_{v \notin S_\infty} \mathcal{O}_v \times \mathcal{P}$$

where S_∞ is the set of archimedean places and

$$\mathcal{P} = \left\{ \sum_{i=1}^{n} t_i e_i \;\middle|\; 0 \leqslant t_i < 1 \right\}$$

with $n = [K : \mathbf{Q}]$, identifying $\prod_{v \in S_\infty} K_v$ with $\mathbf{R}^n$. □

1.2. Duality for locally compact abelian groups: a toolbox.

DEFINITION 1.2. The *Pontryagin dual* of a locally compact abelian group G is the group $\widehat{G}$ of continuous homomorphisms from G to $S^1 \subset \mathbf{C}^*$, called *(unitary) characters*,

$$\mathrm{Hom}_{\text{top.gp.}}(G, S^1)$$

equipped with the compact-open topology, generated by the set

$$\{\chi \in \widehat{G} \mid \chi(B) \subset U\}$$

for every compact $B \subset G$ and open $U \subset S^1$.

PROPOSITION 1.2. *The Pontryagin dual defines an* exact *contravariant functor from the category of locally compact abelian group to itself.*

THEOREM 1.1. *Let G be a locally compact abelian topological group.*

The natural map

$$G \to \widehat{\widehat{G}}$$

is an isomorphism of topological groups.

DEFINITION 1.3 (Abstract Fourier transform). Let f be an L^1-function on G.

The Fourier transform of f is the function

$$\widehat{f} : \widehat{G} \to \mathbf{C}$$

defined by the formula

$$\widehat{f}(\chi) = \int_G f(y)\chi(y)\mathrm{d}y$$

for all character $\chi \in \widehat{G}$.

THEOREM 1.2 (Fourier inversion formula). *Let G be a locally compact and commutative topological group.*

There exists a Haar mesure on the dual $\widehat{G}$ of G, which we denote by $\mathrm{d}\chi$, such that for every L^1 function on G whose Fourier transform is also L^1, we have

$$f(y) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(y)\mathrm{d}\chi$$

for all $y \in G$.

DEFINITION 1.4. If H is a subgroup of G, let

$$H^\perp = \{\chi \in \widehat{G} \mid \chi_{|H} \equiv 1\},$$

called the *orthogonal of H*.

PROPOSITION 1.3. *There* $(G_v)_v$ *be a family of locally compact abelian groups and* $(H_v)_v$ *a family of subgroups that are open compact for all but finitely many* v.

Then we have a canonical isomorphism of topological groups

$$\widehat{\prod{}'(G_v, H_v)} \longrightarrow \prod{}'(\widehat{G_v}, \widehat{G_v/H_v}).$$

1.3. Additive characters. From now on we take G to be a local field K_v seen as an additive topological group.

DEFINITION 1.5. If v is a non-archimedean place, the *conductor* of a non-trivial additive character ψ is the smallest integer m such that $\psi_{\mathfrak{p}^m} \equiv 1$. Sometimes the ideal $\mathfrak{p}^m$ itself is also called *conductor* of ψ.

It is well-defined because ψ is continuous and takes value one at zero.

THEOREM 1.3. *Fix an additive character* $\psi : K_v \to S^1$. *The map*

$$\begin{array}{rcl} \Psi_v : K_v & \longrightarrow & \widehat{K_v} \\ x & \longmapsto & (y \mapsto \psi(xy)) \end{array}$$

induces an isomorphism of locally compact abelian groups.

PROOF. There are three steps.

(1) It is easy to see that Ψ is injective.

(2) To show that Ψ induces an homeomorphism onto its image, we have to compare two topologies on K_v: the original one and the one coming from the subspace topology of $\Psi(K_v)$ pull-backed via Ψ.

Since we are working with topological groups, it is enough to work with a basis of neighborhood of $0 \in K_v$. For the first topology, these are the open balls $B(0, \delta)$ centered in zero with given radius δ, while for the second topology, a basis is given by the sets

$$\{x \in K_v \mid \psi(xB) \subset U\} = \{x \in K_v \mid xB \subset \psi^{-1}(U)\}$$

where B is compact and U is an open subset of S^1 containing 1.

(a) Given such a pair B and U, $\psi^{-1}(U)$ contains a certain $B(0, \delta)$ and B is bounded, it suffices to take δ small enough to have

$$B(0, \delta) \subset \{x \in K_v \mid xB \subset \psi^{-1}(U)\}.$$

(b) Now, given $\delta > 0$, take an $y \in K_v$ with non-trivial image by ψ, take $U \ni 1$ small enough so that $\psi(b) \notin U$ and take B to be a closed disk centered at 0 of radius at leat $|y|/\delta$, so that

$$\{x \in K_v \mid xB \subset \psi^{-1}(U)\} \subset B(0, \delta).$$

(3) Finally, one shows that $\Psi(K_v) = K_v$. First, K_v being complete, $\Psi(K_v)$ is a closed subgroup of $\widehat{K_v}$. Now one can show that the orthogonal of $\Psi(K_v)$ is $\{0\}$, which means that $\Psi(K_v) = \widehat{K_v}$.

□

Actually there is a standard choice for ψ.

DEFINITION 1.6 (Standard local character). We define a standard character ψ on K_v, of conductor zero in case v is non-archimedean, as follows:

- if $K_v = \mathbf{R}$,
$$\psi : x \longmapsto e^{-2\pi i x}$$
(the minus sign has its importance!)
- if $K_v = \mathbf{Q}_p$,
$$\psi : \mathbf{Q}_p \to \mathbf{Q}_p/\mathbf{Z}_p \simeq \mathbf{Z}[1/p]/\mathbf{Z} \hookrightarrow \mathbf{R}/\mathbf{Z}$$
or in other words $\psi_{|\mathbf{Z}_p} \equiv 1$ and $\psi(p^{-n}) = e^{2\pi i/p^n}$ for every $n \geqslant 1$
- if $K_v = \mathbf{F}_p((t))$,
$$\psi : \sum_{-\infty \ll i}^{\infty} c_i t^i \longmapsto e^{2i\pi c_{-1}}$$
- if K_v is either $\mathbf{R}$, $\mathbf{Q}_p$ or $\mathbf{F}_p((t))$ as before, and L_w is a finite separable extension of K_v, then take
$$\psi \circ \mathrm{Tr}_{L_w/K_v}$$
on L_w.

EXAMPLE 1.1. When $K_v = \mathbf{C}$, consider the $\mathbf{R}$-basis of $\mathbf{C}$ given by $(1, i)$, then the matrix of multiplication by $z = x + iy \in \mathbf{C}$ in this basis is
$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$
so that $\mathrm{Tr}_{\mathbf{C}/\mathbf{R}}(z) = 2x = z + \overline{z}$ and our local standard character is
$$z \in \mathbf{C} \mapsto e^{-2\pi i(z+\overline{z})}.$$

See the exercise sheet for a careful study of these standard characters.

1.4. Schwartz-Bruhat functions and their Fourier transforms. We now introduce an important class of functions which will quickly become our best friends.

1.4.1. *Starting local.*

DEFINITION 1.7 (Local Schwartz-Bruhat function). If v is a non-archimedean place, a *local Schwartz-Bruhat function* is a function $f : K_v \to \mathbf{C}$ which is locally constant and compactly supported.

If v is archimedean, a *local Schwartz-Bruhat function* is a smooth function whose derivatives decrease quicker than polynomially.

In both cases we denote by
$$\mathcal{S}(K_v)$$
the corresponding space.

As an exercise, one can show that if K_v is non-archimedean, then any Schwartz-Bruhat function is a finite linear combination of characteristic functions of open balls.

This class of functions (named after Laurent Schwartz and François Bruhat) behaves very well with respect to Fourier transforms: the Fourier transform of a Schwartz-Bruhat function is always defined and is still a Schwartz-Bruhat function.

DEFINITION 1.8. If f is a Schwartz-Bruhat function on K_v, the Fourier transform of f is the function defined by

$$\widehat{f}(y) = \int_{K_v} f(x)\psi(xy)\mathrm{d}x$$

for all $y \in K_v$.

As an exercise, one can compute the Fourier transform of the characteristic function of a ball.

From now on we fix once and for all the Haar measure so that it is self-dual relative to the standard additive character ψ.

- if $K_v = \mathbf{R}$ then we take the standard Lebesgue measure
- if $K_v = \mathbf{C}$ then we take twice the standard Lebesgue measure: $\mathrm{d}z\mathrm{d}\overline{z} = 2\mathrm{d}x\mathrm{d}y$
- if K_v is non-archimedian then we normalize $\mathrm{d}x$ so that $\mathcal{O}_v$ has measure

  $$1/\sqrt{|\mathcal{O}_v/\mathcal{D}_v|}$$

  where $\mathcal{D}_v = \mathfrak{m}_v^d\mathcal{O}_v$ is the different of K_v, where d is the largest integer such that $\mathrm{Tr}_{K_v/F_v}(\mathfrak{m}_v^{-d}\mathcal{O}_v) \subset \mathcal{O}_v$ (where F_v is either $\mathbf{Q}_p$ or $\mathbf{F}_q((t))$ depending on the characteristic).

PROPOSITION 1.1 (Fourier inversion formula). *For any Schwartz-Bruhat function f on K_v the relation*

$$f(x) = \int_{K_v} \widehat{f}(y)\overline{\psi(xy)}\mathrm{d}y = \widehat{\widehat{f}}(-x)$$

holds for all $x \in K_v$.

PROOF. Since any Schwartz-Bruhat function is a finite linear combination of characteristic functions of balls, it suffices to prove the relation for such elementary functions. Do it! $\square$

1.4.2. *Going global.*

DEFINITION 1.9 (Global Schwartz-Bruhat function). The space of global Schwartz-Bruhat functions is the space of functions on the space of adèles $\mathbb{A}_K$ given by the limit

$$\bigotimes_{v\in M_K}{}' \mathcal{S}(K_v) = \varinjlim_{\substack{S\subset M_K \\ \text{finite}}} \bigotimes_{v\in S} \mathcal{S}(K_v).$$

More concretely, a global Schwartz-Bruhat function can be written

$$f = \prod_v f_v$$

with $f_v = \mathbf{1}_{\mathcal{O}_v}$ for all places v of K outside of a finite set S, while for places $v \in S$, the local function f_v is a local Schwartz-Bruhat function.

Definition 1.10 (Standard global character). If K is a number field, for every place v of K, let ψ_v be the standard local character on K_v.

If K is the function field of a curve, pick a global meromorphic 1-form ω, and set for every place

$$\psi_v : x \in K_v \mapsto \exp\left(\frac{2\pi i}{p} \operatorname{Tr}_{\kappa_v/\mathbf{F}_p} \operatorname{Res}_v(x\omega)\right)$$

where

$$\operatorname{Res}_v : \begin{cases} k_v((u))\mathrm{d}u & \longrightarrow & k_v \\ \sum_i c_i u^i \mathrm{d}u & \longmapsto & c_{-1} \end{cases}$$

does not depend on the choice of the local parameter u.

In both cases, the standard global character ψ_K on $\mathbb{A}_K$ is given by the product

$$\psi_K = \prod_v \psi_v.$$

Example 1.2 (Important in some applications). Assume that K is the function field of $\mathbf{P}^1_{\mathbf{F}_p}$, and that we choose a local parameter, so that $K = \mathbf{F}_p(t)$ and $\omega = \mathrm{d}t$. Then the completion of K at the place corresponding to the point at infinity is $K_\infty = \mathbf{F}_p((t^{-1}))$ and a local parameter around infinity is given by $u = 1/t$, so that

$$\omega = \mathrm{d}t = -1/u^2 \mathrm{d}u$$

and

$$\psi_\infty\left(\sum_{i \ll \infty} c_i t^i\right) = \operatorname{Res}_\infty\left(\sum_{i \ll \infty} c_i t^i \omega\right) = \operatorname{Res}_\infty\left(-\sum_{\infty \ll i} c_{-i+2} u^i \mathrm{d}u\right) = -c_{-1}$$

so one gets something different from the definition of local character we gave earlier.

Proposition 1.4. *The standard global character on $\mathbb{A}_K$ is trivial on K.*

Proof. Postponed. □

Proposition 1.2. *The Pontryagin dual of $\mathbb{A}_K$ is $\mathbb{A}_K$ itself, with an isomorphism being given by*

$$x \in \mathbb{A}_K \mapsto \psi_x = (y \mapsto \psi_K(xy)) \in \widehat{\mathbb{A}_K}.$$

Proof. We saw earlier that $\widehat{\mathbb{A}_K}$ can be identified with

$$\prod{}' (\widehat{K_v}, \widehat{K_v/\mathcal{O}_v})$$

so Ψ is just $\prod_v \Psi_v$ where Ψ_v is the isomorphism given by Theorem 1.3. □

Proposition 1.5. *Under the previous isomorphism, the Pontryagin dual of $\mathbb{A}_K/K$ is isomorphic to K.*

Proof. Pontryagin duality sends compact groups to discrete groups and conversely. Hence $K^\perp$, as the dual of the compact group $\mathbb{A}_K/K$, is discrete. It is actually a K-subspace of $\widehat{\mathbb{A}_K} \simeq \mathbb{A}_K$ via $x \mapsto \psi_x$, so $K^\perp/K$ is a discrete subgroup of the compact $\mathbb{A}/K$, hence is finite, but it is as well a vector space above the infinite field K, hence $K^\perp/K = 0$ and finally $K^\perp = K$. □

G	$\widehat{G}$
K_v (local field)	K_v
$\mathbf{Z}$	$\mathbf{R}/\mathbf{Z}$
$\mathbf{Z}_p$	$\mathbf{Q}_p/\mathbf{Z}_p$
$\mathbf{A}_K$	$\mathbf{A}_K$
K (global field)	$\mathbf{A}_K/K$
finite	finite
discrete	compact
compact	discrete
loc. compact	loc. compact
discrete torsion	profinite

DEFINITION 1.11 (Tamagawa measure). The Tamagawa measure on $\mathbb{A}_K$ is the Haar measure

$$\mathrm{d}x = \prod_v \mathrm{d}x_v.$$

defined by

$$\int_U \mathrm{d}x = \prod_v \int_{U_v} \mathrm{d}x_v$$

for every basic open subset $U = \prod_v U_v$ of $\mathbb{A}_K$.

DEFINITION 1.12 (Global or adelic Fourier transform). For any integrable function f its Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(y) = \int_{\mathbb{A}_K} f(x)\psi(xy)\mathrm{d}x$$

for every $y \in \mathbb{A}_K$.

LEMMA 1.2 (Invariance under translation by a compact subgroup). *Postponed.*

1.5. Poisson formula in the adelic setting.

THEOREM 1.4 (Poisson formula). *Let f be a function on $\mathbb{A}_K$ such that both f and $\widehat{f}$ are integrable and that*

$$\sum_{\gamma\in K} f(\gamma + x)$$

converges absolutely and uniformly when x belongs to any compact subset of $\mathbb{A}_K/K$.

Then

$$\sum_{\gamma\in K} f(\gamma + x) = \sum_{\gamma\in K} \widehat{f}(\gamma + x)$$

for every adèle $x \in \mathbb{A}_K$. In particular,

$$\sum_{\gamma\in K} f(\gamma) = \sum_{\gamma\in K} \widehat{f}(\gamma).$$

PROOF. It is an application of the Fourier inversion formula to the average function

$$F : x \in \mathbb{A}_K \mapsto \sum_{\gamma\in K} f(\gamma + x) \in \mathbf{C}.$$

Since this is a K-periodic function, it induces a function on the quotient $\mathbb{A}_K/K$ which we denote again by F. The Pontryagin dual of $\mathbb{A}_K/K$ is isomorphic to K and we compute the Fourier transform of F at $\gamma \in K$ to get:

$$\begin{aligned}
\widehat{F}(\gamma) &= \int_{\mathbb{A}_K/K} F(\overline{x})\psi(\gamma\overline{x})\mathrm{d}\overline{x} \\
&= \int_D \sum_{\ell\in K} f(x+\ell)\psi(\gamma x)\mathrm{d}x \\
&= \sum_{\ell\in K}\int_D f(x+\ell)\psi(\gamma x)\mathrm{d}x \\
&= \sum_{\ell\in K}\int_{D+\ell} f(y)\,\underbrace{\psi(\gamma(y-\ell))}_{=\psi(\gamma y)}\,\mathrm{d}y \\
&= \int_{\mathbb{A}_K} f(y)\psi(\gamma y)\mathrm{d}z \\
&= \widehat{f}(\gamma).
\end{aligned}$$

Now the Fourier inversion formula gives

$$\begin{aligned}
F(\overline{x}) &= \sum_{\gamma\in K}\widehat{F}(\gamma)\overline{\psi(\gamma\overline{x})} \\
&= \sum_{\gamma\in K}\widehat{f}(\gamma)\overline{\psi(\gamma x)} \\
&= \sum_{\gamma\in K}\widehat{f}(\gamma+x)
\end{aligned}$$

that is to say

$$\sum_{\gamma\in K} f(\gamma+x) = \sum_{\gamma\in K}\widehat{f}(\gamma+x)$$

as wanted. □

1.6. A classical application: Riemann-Roch for curves over finite fields. In this subsection the field K is the function field of a smooth projective curve $\mathscr{C}$ defined above a finite field:

$$K = \mathbf{F}_q(\mathscr{C}).$$

DEFINITION 1.13. For any divisor $D = \sum_v d_v[v]$ on the curve $\mathscr{C}$, let

$$\mathbb{A}_K(D) = \{(x_v) \in \mathbb{A}_K \mid v(x_v) + d_v \geqslant 0\}$$

and

$$K(D) = \mathbb{A}_K(D) \cap K.$$

LEMMA 1.3. *The set $K(D)$ admits a structure of finite dimensional vector space over $\mathbf{F}_q$.*

We denote by $\ell(D)$ its dimension.

THEOREM 1.5 (Riemann-Roch for $\mathbf{F}_q$-curves). *For any divisor D on the curve $\mathscr{C}$ and any meromorphic form $\omega \in \Omega_K$,*

$$\ell(D) - \ell(\mathcal{K} - D) = \deg(D) + 1 - g(\mathscr{C})$$

where $\mathcal{K} = \operatorname{div}(\omega)$.

PROOF. Apply the Poisson formula (with respect to the standard character on $\mathbf{A}_K$ attached to ω) to $f = \mathbf{1}_{\mathbf{A}_K(D)}$. □

1.7. The adelic space of an algebraic variety.

DEFINITION 1.14. As a set, the adelic space of V is the set

$$V(\mathbb{A}_K)$$

of its adelic points:

$$\left\{ (x_v)_{v \in M_K} \in \prod_{v \in M_K} V(K_v) \;\middle|\; x_v \in V(\mathcal{O}_v) \text{ for almost every } v \in M_K \right\}.$$

REMARK 1.1. If V is complete, then $V(\mathcal{O}_v) = V(K_v)$ for every $v \in M_K$ and the adelic space of V is just

$$V(\mathbb{A}_K) = \prod_{v \in M_K} V(K_v)$$

2. Counting of rational points on some equivariant compactifications of vector spaces

2.1. Models and associated metrics.

DEFINITION 2.1 (Metric on a line bundle). A *metric* on a line bundle $L \to M$ is the datum for every open subset U of M of

$$\|\cdot\| : \ell \in \Gamma(U, L) \longmapsto (\|\ell\| : U \to \mathbf{R}_+)$$

such that

(1) $\|\cdot\|$ is continuous;
(2) for every $\ell \in \Gamma(U, L)$ and $x \in U$, $\|\ell\|$ is positive if and only if $\ell(x) \neq 0$;
(3) it is compatible with restriction to any open subset V inside U;
(4) $\|f\ell\|(x) = |f(x)|\|\ell\|(x)$ for every K-analytic function on U.

The datum of a line bundle together with a metric on it is called a *metrized line bundle*.

This notion can easily be extended to vector bundles.

In this paragraph we explain a very important construction of a certain metric that is going to be used in every chapter of this course.

CONSTRUCTION 2.1 (Metric induced by a model). Assume that K is a non-Archimedean local field (not necessarily complete), with valuation field R, and that $\mathscr{X}$ is a flat separated R-scheme of finite type, with smooth generic fibre $X = \mathscr{X} \otimes_R K$, so that $\mathscr{X}(R)$ injects into $X(K)$ as a compact subset.

Consider $\mathscr{L}$ a coherent sheaf on $\mathscr{X}$ whose generic fibre L is a line bundle (we say that $\mathscr{L}$ is a model of L). This line bundle induces a line bundle on the K-analytic manifold $M = \mathscr{X}(R)$, again denoted by L.

If $x \in M = \mathscr{X}(R)$, the fibre $x^*\mathscr{L}$ is an R-module of finite type, which possibly has a non-empty torsion part $x^*\mathscr{L}_{\mathrm{tors}}$, and x_L^*L is a one dimensional K-vector space.

$$\begin{tikzcd}
x_K^*L \arrow[r] \arrow[dd] & x^*\mathscr{L} \arrow[dd] & \mathscr{L} \arrow[d, dash] \\
 & & \mathscr{X} \arrow[d] \\
\operatorname{Spec}(K) \arrow[r] \arrow[urr, "x_K"] & \operatorname{Spec}(R) \arrow[r, equal] \arrow[ur, "x"'] & \operatorname{Spec}(R)
\end{tikzcd}$$

Then, the R module

$$\mathscr{L}(x) = x^*\mathscr{L}/x^*\mathscr{L}_{\mathrm{tors}}$$

can be seen as a lattice inside x_K^*L: indeed, since the square is Cartesian, a point of $x^*\mathscr{L}$, seen as a section $\operatorname{Spec}(R) \to x^*\mathscr{L}$, then composed with $\operatorname{Spec}(K) \to \operatorname{Spec}(R)$ induces a unique K-point of x_K^*L.

Given any generator y_0 of this lattice, we obtain a norm on the K-vector space x_K^*L by setting $\|ay_0\| = |a|$ for all $a \in K$ (this does not depend on the choice of y_0 since two generators differ by an invertible element).

Now given any section s of L on an open subset U of M, we set $\|s\|(x) = \|s(x)\|$ for all $x \in U$.

PROPOSITION 2.1. *The previous construction defines a metric on the line bundle L in the sense of Definition 2.1.*

DEFINITION 2.2. An *adelic metric* on a vector bundle, on a variety defined above a global field, is a collection of metrics above each completion of the global field, that coincide with the metrics induced by a model except at a finite number of places.

When blowing-up, one wishes to build compatible metrics between the variety one started with and its blow-up, as follows. Let S be the spectrum of a Dedekind ring or the specturm of a valued field. Let X be a quasi-projective flat scheme over S, $\mathscr{I}$ a sheaf of ideal on X, and $Z = V(\mathscr{I})$. Let $\pi : Y \to X$ be the blow-up of $V(\mathscr{I})$, that is to say $Y = \mathbf{Proj}_X(\oplus_n \mathscr{I}^n)$. The inverse image of Z is a Cartier divisor D, the invertible sheaf $\mathscr{O}_Y(D)$ admits a canonical section s_D obtained by pulling-back the canonical inclusion $\mathscr{I} \hookrightarrow \mathscr{O}_X$ and $\mathscr{O}_Y(-D) = \mathscr{I} \cdot \mathscr{O}_Y$.

(1) First, choose a locally free sheaf $\mathscr{E}$ of finite rank on X with a global section σ_Z such that $\operatorname{div}(\sigma_Z) = Z$. This induces a **surjective homomorphism**

$$\phi : \left\{ \begin{array}{rcl} \mathscr{E}^\vee = Hom(\mathscr{E}, \mathscr{O}_X) & \longrightarrow & \mathscr{I} \\ f & \longmapsto & f \circ \sigma_Z \end{array} \right.$$

from which one hence builds a closed immersion

$$Y = \mathbf{Proj}_X(\oplus_n \mathscr{I}^n) \quad \hookrightarrow \quad \mathbf{P}(\mathscr{E}^\vee) = \mathbf{Proj}_X(\oplus_n Sym^n_X(\mathscr{E}^\vee))$$

such that

$$\mathscr{O}_Y(-D) = \mathscr{I} \cdot \mathscr{O}_Y = \mathscr{O}_{\mathbf{P}(\mathscr{E}^\vee)}(1)_{|Y}.$$

(2) Above Y, the universal quotient map

$$\pi^*\mathscr{E}^\vee \to \mathscr{O}_{\mathbf{P}(\mathscr{E}^\vee)}(1)$$

is given by $\pi^*\phi$. It allows one to define a quotient metric: for any local section s of $\mathscr{O}_{\mathbf{P}(\mathscr{E}^\vee)}(1)$,

$$\|s\|(y) = \inf_{\substack{t \text{ local section} \\ (\pi^*\phi)(t)=s}} \|t\|(y).$$

(3) Restricting this to Y gives a norm on $\mathscr{O}_Y(-D)$. The dual norm on $\mathscr{O}_Y(D)$ of s_D is given by

$$\|s_D\|(y) = \sup_{s\neq 0} \frac{|\langle s_D, s\rangle|}{\|s\|} = \sup_{t\neq 0} \frac{|\langle s_D, \pi^*\phi(t)\rangle|}{\|t\|}.$$

(4) Away from D on Y, a local section of $\pi^*\mathscr{E}^\vee$ is nothing else than a local section t of $\mathscr{E}^\vee$, and

$$\langle s_D, \pi^*\phi(t)\rangle = \langle \pi^*(\mathscr{I} \hookrightarrow \mathscr{O}_X), \pi^*\phi(t)\rangle = t \circ \sigma_Z$$

locally, thus

$$\|s_D\| = |\phi| = \|\sigma_Z\|$$

by definition of the dual norm on $\mathscr{E}^\vee$, viewing $\|\cdot\|$ as

$$\|\cdot\| : \mathscr{E}^\vee \to \mathscr{O}_X.$$

THEOREM 2.1. *Let X be an algebraic variety, $\mathscr{I} \subset \mathscr{O}_X$ a sheaf of ideals on X and $\pi : Y \to X$ the blow-up of $V(\mathscr{I})$. with D being the exceptional divisor. Let $\mathscr{E}$ be a locally free sheaf of finite rank on X and σ_Z a global section of $\mathscr{E}$ such that $V(\mathscr{I}) = \mathrm{div}(\sigma_Z)$.*

Assume that $\mathscr{E}$ is metrized. Then $\mathscr{O}_Y(D)$ admits a canonical metric such that

$$\|s_D\|(y) = \|\sigma_Z\|(\pi(y))$$

for every $y \in Y$ (s_D is the canonical section of $\mathscr{O}_Y(D)$ obtained by pulling-back the canonical inclusion $\mathscr{I} \hookrightarrow \mathscr{O}_X$).

In practice, we take $\mathscr{E}$ to be

$$\mathscr{E} = \mathscr{L}_1 \oplus ... \oplus \mathscr{L}_r$$

together with sections s_i of the line bundles $\mathscr{L}_i$ so that $Z = \cap \mathrm{div}(s_i)$. Assume that the $\mathscr{L}_i$'s are metrized: one can endowed $\mathscr{E}$ with the Hermitian (archimedean case) or sup metric (non-archimedean case) associated to the metrics on the $\mathscr{L}_i$'s. By the previous theorem, $\mathscr{O}_Y(D)$ can be canonically metrized so that

$$\|s_D\|^2(y) = \|\sigma_Z\|^2(\pi(y)) = \sum_{i=1}^r \|s_i\|^2(\pi(y))$$

respectively

$$\|s_D\|(y) = \|\sigma_Z\|(\pi(y)) = \max_{i=1,\dots,r} \|s_i\|(\pi(y)).$$

EXAMPLE 2.1. Assume that $X = \mathbf{P}^n$, $\mathscr{L}_i = \mathscr{O}_{\mathbf{P}^n}(n_i)$ and s_i corresponds to g_i homogeneous of degree n_i, and that $\pi(y) = (x_0 : ... : x_n)$. Then

$$\|s_D\|^2(y) = \sum_{i=1}^{r} \frac{|g_i(x_0, ..., x_n)|^2}{\left(\sum_{j=0}^{n} |x_j|^2\right)^{n_i}}$$

respectively

$$\|s_D\|(y) = \max_{i=1,...,r} \frac{|g_i(x_0, ..., x_n)|}{(\max_{j=1,...,n} |x_j|)^{n_i}}$$

Assume now that Z is an integral divisor contained *inside* $Z_0 = \{x_0 = 0\}$. Then its ideal is of the form $(x_0, f(x_1, ..., x_n))$ with f homogeneous of degree d and for $\pi(y) \notin Z_0$,

$$\|s_D\|^2(y) = \frac{1}{1 + \sum_{j=1}^{n} |x_j|^2} + \frac{|f(x_1, ..., x_n)|^2}{\left(1 + \sum_{j=1}^{n} |x_i|^2\right)^d}$$

over archimedean places and

$$\|s_D\|(y) = \max\left(\frac{1}{\max\left(1, \max_{i=1,...,n} |x_i|\right)}, \frac{\max(1, |f(1_0, ..., x_n)|)}{\max\left(1, \max_{i=1,...,n} |x_i|\right)^d}\right)$$

over non-archimedean places, whenever $\pi(y) = (1 : x_1 : ... : x_n)$.

2.2. A compactification of $\mathbf{G}_a^2$. We are going to use the following family of norms.

$$\boldsymbol{x} \in \mathbf{Q}_p^n \qquad \|\boldsymbol{x}\|_p = \max_{i=1,...,n} |x_i|_p$$

$$\boldsymbol{x} \in \mathbf{R}^n \qquad \|\boldsymbol{x}\|_\infty = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Making use of the adelic techniques we learned so far in this course, in particular, of the additive Poisson formula Theorem 1.4, our goal is to prove the following theorem of Chambert-Loir and Tschinkel [**CLT00**].

THEOREM 2.2. *Let U be the complement in $\mathbf{P}^2_{\mathbf{Q}}$ of $\{x_0 = 0\} \simeq \mathbf{P}^1_{\mathbf{Q}}$ and let $p_1, ..., p_r$ distinct $\mathbf{Q}$-points on this latter line.*

Let X be the blow-up of $\mathbf{P}^2_{\mathbf{Q}}$ at the r points $p_1, ..., p_r$ and $H_{\omega_X^{-1}}$ be the exponential height associated to the metrized line bundle ω_X^{-1}.

Then for every real number $B > 0$ the set

$$\{x \in U(\mathbf{Q}) \mid H_{\omega_X^{-1}} \leqslant B\}$$

is finite and

$$\#\{x \in U(\mathbf{Q}) \mid H_{\omega_X^{-1}} \leqslant B\} \sim \frac{1}{3 \cdot 2^r \cdot r!} \tau(\omega_X) B \cdot \log(B)^r$$

as $B \to \infty$.

REMARK 2.1. Actually, following [**CLT00**] we will prove a stronger result: there exists a polynomial P_X of degree r and leading coefficient equal to $\frac{1}{3 \cdot 2^r \cdot r!}\tau(\omega_X)$ such that

$$\#\{x \in U(\mathbf{Q}) \mid H_{\omega_X^{-1}} \leqslant B\} = BP_X(\log(B)) + O(B^{1-\delta})$$

for some real number $\delta > 0$.

2.3. Heights and zeta function. We work above $K = \mathbf{Q}$. Recall the Definition 2.1 of metrized line bundles page 47.

We assume that $p_1, ..., p_r$ are contained in the line at infinity $Z_0 = \{x_0 = 0\}$. Recall that U is the open complement of Z_0. For all $i \in \{1, ..., r\}$ there is a primitive linear form $\ell_i \in \mathbf{Z}[x_1, x_2]$ such that $p_i = V(x_0, \ell_i)$.

The local height with respect to D_i for $i \in \{1, ..., r\}$ is given by

$$H_{i,p}(\boldsymbol{x}) = \frac{\max(1, \|\boldsymbol{x}\|_p)}{\max(1, |\ell_i(\boldsymbol{x})|_p)}$$

at a finite place p and

$$H_{i,\infty}(\boldsymbol{x}) = \sqrt{\frac{1 + \|\boldsymbol{x}\|_\infty^2}{1 + |\ell_i(\boldsymbol{x})|_p^2}}$$

at the Archimedean place at ∞. For the remaining divisor D_0,

$$H_{0,p} = \max(1, \|\boldsymbol{x}\|_p) \prod_{i=1}^{r} H_{i,p}^{-1}(\boldsymbol{x})$$

and

$$H_{0,\infty} = \sqrt{1 + \|\boldsymbol{x}\|_p^2} \prod_{i=1}^{r} H_{i,\infty}^{-1}(\boldsymbol{x}).$$

Definition 2.3. The global height corresponding to the complexified line bundle

$$D(\boldsymbol{s}) = \otimes_{i=0}^{r} D_i^{\otimes s_i}$$

is defined by

$$H(\boldsymbol{s}; \boldsymbol{x}) = \prod_{i=0}^{r} H_i(\boldsymbol{x})^{s_i}.$$

This gives a pairing

$$H : \mathrm{Pic}^G(X)_{\mathbf{C}} \times \mathbf{G}_a^2(\mathbb{A}_{\mathbf{Q}}) \to \mathbf{C}^*$$

which is invariant under the action of $\prod_p \mathbf{G}_a^2(\mathbf{Q}_p)$.

The corresponding height zeta function is

$$\mathcal{Z}(\boldsymbol{s}) = \sum_{\boldsymbol{x} \in \mathbf{G}_a^2(\mathbf{Q})} H(\boldsymbol{s}; \boldsymbol{x})^{-1}.$$

2.4. Applying the Poisson formula. Fourier transform of the height: for every $\psi \in \mathbf{G}_a^n(\mathbb{A}_K)$

$$\widehat{H}(\boldsymbol{s}; \psi) = \int_{\mathbf{G}_a^n(\mathbb{A}_K)} H(\boldsymbol{s}; \boldsymbol{x}) \psi(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

whenever this integral converges.

Lemma 2.1. *If ψ is non-trivial on the compact subgroup $\mathbf{K} = \prod_v \mathcal{O}_v^n$ then*

$$\widehat{H}(\boldsymbol{s}; \psi) = 0.$$

PROOF. Since $\boldsymbol{x} \mapsto H(\boldsymbol{s};\boldsymbol{x})$ is invariant under translation by elements of $\mathbf{K}$, if ψ is non-trivial at $\boldsymbol{a} \in \mathbf{K} \setminus \{\mathbf{0}\}$, applying a change of variable $\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{a}$ one gets

$$\widehat{H}(\boldsymbol{s};\psi) = \int_{\mathbf{G}_a^n(\mathbb{A}_K)} H(\boldsymbol{s};\boldsymbol{x})\psi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \underbrace{\psi(\boldsymbol{a})}_{\neq 1} \int_{\mathbf{G}_a^n(\mathbb{A}_K)} H(\boldsymbol{s};\boldsymbol{x})\psi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

hence $\widehat{H}(\boldsymbol{s};\psi) = 0$. □

PROPOSITION 2.2. *The height zeta function can be rewritten*

$$\mathcal{Z}(\boldsymbol{s}) = \sum_{\boldsymbol{a} \in \mathbf{Z}^2} \widehat{H}(\boldsymbol{s}, \psi_{\boldsymbol{a}})$$

where

$$\widehat{H}(\boldsymbol{s}, \psi_{\boldsymbol{a}}) = \widehat{H}_\infty(\boldsymbol{s}, \psi_{\boldsymbol{a}}) \prod_{p \text{ prime}} \widehat{H}_p(\boldsymbol{s}, \psi_{\boldsymbol{a}}).$$

PROOF. Applying Theorem 1.4 to

$$\sum_{\boldsymbol{x} \in \mathbf{Q}^2} H(\boldsymbol{s};\boldsymbol{x})^{-1}.$$

one gets

$$Z(\boldsymbol{s}) = \sum_{\boldsymbol{a} \in \widehat{\mathbf{Q}^2}} \widehat{H}(\boldsymbol{s}, \psi_{\boldsymbol{a}}).$$

But the invariance by $\mathbf{K}$ gives that the sum is actually on $\mathbf{Z}^2$. □

From now on, we identify $\boldsymbol{a} \in \mathbf{Z}^2$ with the linear form

$$\langle \cdot, \boldsymbol{a} \rangle \in \mathrm{Hom}_{\mathrm{gp}}(\mathbf{G}_a^2, \mathbf{G}_a)(\mathbf{Q})$$

it defines on $\mathbf{G}_a^2$, and with $\psi_{\boldsymbol{a}} = \psi(\langle \cdot, \boldsymbol{a} \rangle)$.

DEFINITION 2.4. We say that a non-trivial character $\boldsymbol{a} \in \mathbf{Z}^2$ is *generic* is $\psi_{\boldsymbol{a}}$ is not proportional to any of the ℓ_i's.

We say that a non-trivial character is *special* if it is proportional to some ℓ_i (necessarily, this holds for a unique i).

In the first case, let

$$S(\boldsymbol{a}) = \{p \in S \mid p \text{ divides } \det(\ell_j, \boldsymbol{a}) \text{ for some } j \in \{1, ..., r\}\}.$$

In the second case, let

$$S(\boldsymbol{a}) = \{p \in S \mid p \text{ divides } \det(\ell_j, \boldsymbol{a}) \text{ for some } j \neq i\}.$$

Note that if p divides $\boldsymbol{a}$, then $p \in S(\boldsymbol{a})$.

It may be convenient to set $S(\mathbf{0}) = S$.

We end up with a decomposition:

$$\sum_{\boldsymbol{a} \in \mathbf{Z}^2} \widehat{H}(\boldsymbol{s}, \psi_{\boldsymbol{a}}) = \widehat{H}(\boldsymbol{s}, \psi_{\mathbf{0}}) + \sum_{\substack{\boldsymbol{a} \in \mathbf{Z}^2 \setminus \{\mathbf{0}\} \\ \boldsymbol{a} \text{ generic}}} \widehat{H}(\boldsymbol{s}, \psi_{\boldsymbol{a}}) + \sum_{i=1}^{r} \sum_{\substack{\boldsymbol{a} \in \mathbf{Z}^2 \setminus \{\mathbf{0}\} \\ \boldsymbol{a} \text{ special for } \ell_i}} \widehat{H}(\boldsymbol{s}, \psi_{\boldsymbol{a}}).$$

2.5. Analysis of the terms at good reductions. Let p be a prime not in S. Let us introduction an useful decomposition of $\mathbf{Q}_p^2$.

- $U(0) = \mathbf{Z}_p^2$ on which all the H_i's are equal to 1;
- $U_i(\alpha,\beta)$ for $i \in \{1,...,r\}$ and $1 \leqslant \beta < \alpha$, is the set of $\boldsymbol{x} \in \mathbf{Q}_p$ such that
$$\|\boldsymbol{x}\| = p^\alpha \qquad |\ell_i(\boldsymbol{x})| = p^{\alpha-\beta}$$
so that
$$H_i(\boldsymbol{x}) = p^\beta \qquad H_j(\boldsymbol{x}) = 1 \text{ for } j \neq i, 0 \qquad H_0(\boldsymbol{x}) = p^{\alpha-\beta};$$
- $U_i(\alpha)$ for $i \in \{1,...,r\}$ and $\alpha \geqslant 1$ is the set of $\boldsymbol{x} \in \mathbf{Q}_p$ such that
$$\|\boldsymbol{x}\| = p^\alpha \qquad |\ell_i(\boldsymbol{x})| \leqslant 1$$
so that
$$H_i(\boldsymbol{x}) = p^\alpha \qquad H_j(\boldsymbol{x}) = 1 \text{ for } j \neq i;$$
- $U_0(\alpha)$ for $1 \leqslant \alpha$, is the set of $\boldsymbol{x} \in \mathbf{Q}_p^2$ such that
$$\|\boldsymbol{x}\| = p^\alpha \qquad |\ell_j(\boldsymbol{x})| = p^\alpha \text{ for } j \in \{1,...,r\}$$
so that
$$H_0(\boldsymbol{x}) = p^\alpha \qquad H_j(\boldsymbol{x}) = 1 \text{ for } j \in \{1,...,r\}.$$

	$U(0)$	$U_i(\alpha,\beta)$	$U_i(\alpha)$	$U_0(\alpha)$
$\|\boldsymbol{x}\|$	1	p^α	p^α	p^α
$\lvert\ell_i(\boldsymbol{x})\rvert$	1	$p^{\alpha-\beta}$	$\leqslant 1$	p^α
$H_0(\boldsymbol{x})$	1	$p^{\alpha-\beta}$	1	p^α
$H_i(\boldsymbol{x})$	1	p^β	p^α	1
$H_j(\boldsymbol{x})\, j \neq i$	1	1	1	1

In particular, the decomposition above shows that our local height function, for $p \notin S$, is a finite linear combination of Schwartz-Bruhat functions.

2.5.1. *Trivial character.* First, let us compute the volumes of the subsets we just introduced.

LEMMA 2.2. *For α and β as above, we have the following table of p-adic volumes.*

	$U(0)$	$U_i(\alpha,\beta)$	$U_i(\alpha)$	$U_0(\alpha)$
Vol	1	$p^{2\alpha-\beta}\frac{(p-1)^2}{p^2}$	$p^\alpha\frac{p-1}{p}$	$p^{2\alpha}\frac{(p-1)(p+1-r)}{p^2}$.

PROOF. We can always assume that $\ell_i(\boldsymbol{x}) = x_1$. Then
$$U_i(\alpha,\beta) = p^{\beta-\alpha}\mathbf{Z}_p^\times \times p^{-\alpha}\mathbf{Z}_p^\times$$
$$U_i(\alpha) = \mathbf{Z}_p \times p^{-\alpha}\mathbf{Z}_p^\times$$
and since $\mathrm{Vol}(p^k\mathbf{Z}_p^\times) = p^{-k}\left(1-\frac{1}{p}\right)$ we get the first two colums. For the third one, remark that in fact $p^\alpha U_0(\alpha)$ is the complement in $\mathbf{Z}_p^2$ of $1+(p-1)r$ disjoint two-dimensional balls, each of these balls having radius p^{-1}, hence the volume of $U_0(\alpha)$ is
$$p^{2\alpha}\left(1-\frac{1+(p-1)r}{p^2}\right) = p^{2\alpha}\frac{(p-1)(p+1-r)}{p^2}$$
hence the lemma. $\square$

Lemma 2.3. *Let $p \notin S$ and $\boldsymbol{s} \in \mathbf{C}^{s+1}$ such that $\Re(s_0) > 2$ and $\Re(s_i) > 1$ for $i \in \{1, ..., r\}$.*
Then

$$\widehat{H}_p(\boldsymbol{s}; \psi_{\mathbf{0}}) = 1 + \frac{p^2 - 1}{p^{s_0} - p^2} + \frac{p-1}{p^{s_0} - p^2} \sum_{i=1}^{r} \frac{p^{s_0 - 1} - p^{s_i - 1}}{p^{s_i - 1} - 1}.$$

Proof. We decompose $\widehat{H}_p(\boldsymbol{s}; \psi_{\mathbf{0}})$ as a sum of integrals over $U(0)$, $U_i(\alpha, \beta)$ and $U(\alpha)$ and first compute the contribution of each of the terms. First,

$$\int_{U_i(\alpha,\beta)} H_p(\boldsymbol{s}; \psi_{\mathbf{0}})^{-1} \mathrm{d}\boldsymbol{x} = \mathrm{Vol}(U_i(\alpha, \beta)) p^{-s_0(\alpha - \beta) - s_i \beta} = \frac{(p-1)^2}{p^2} p^{2\alpha - \beta} p^{-\alpha s_0} p^{-\beta(s_i - s_0)}$$

and if we sum over all $1 \leqslant \beta < \alpha$ we get

$$\begin{aligned}\sum_{1 \leqslant \beta < \alpha} \int_{U_i(\alpha,\beta)} H_p(\boldsymbol{s}; \psi_{\mathbf{0}})^{-1} \mathrm{d}\boldsymbol{x} &= \frac{(p-1)^2}{p^2} \sum_{1 \leqslant \beta < \alpha} p^{-\alpha(s_0 - 2)} p^{-\beta(s_i - s_0 + 1)} \\ &= \frac{(p-1)^2}{p^2} \frac{1}{p^{s_0 - 2} - 1} \frac{1}{p^{s_i - 1} - 1}.\end{aligned}$$

The contribution of the $U_i(\alpha)$ is

$$\frac{p-1}{p} \sum_{\alpha=1}^{\infty} p^{\alpha(1 - s_i)} = \frac{p-1}{p} \frac{1}{p^{s_i - 1} - 1}$$

while the one coming from $U_0(\alpha)$ is

$$\frac{(p-1)(p+1-r)}{p^2} \sum_{\alpha=1}^{\infty} p^{\alpha(2 - s_0)} = \frac{(p-1)(p+1-r)}{p^2} \frac{1}{p^{s_0 - 1} - 1}.$$

Now

$$\begin{aligned}\widehat{H}_p(\boldsymbol{s}; \psi_{\mathbf{0}}) &= 1 + \sum_{i=1}^{r} \frac{(p-1)^2}{p^2} \frac{1}{p^{s_0 - 2} - 1} \frac{1}{p^{s_i - 1} - 1} + \sum_{i=1}^{r} \frac{p-1}{p} \frac{1}{p^{s_i - 1} - 1} \\ &\quad + \frac{(p-1)(p+1-r)}{p^2} \frac{1}{p^{s_0 - 1} - 1} \\ &= ... \\ &= 1 + \frac{p^2 - 1}{p^{s_0} - p^2} + \frac{p-1}{p^{s_0} - p^2} \sum_{i=1}^{r} \frac{p^{s_0 - 1} - p^{s_i - 1}}{p^{s_i - 1} - 1}.\end{aligned}$$

□

2.5.2. *Generic characters.*

Lemma 2.4. *Let $\boldsymbol{a}$ be a generic character and $p \notin S(\boldsymbol{a})$. We have the following table.*

U	$U(0)$	$U_i(\alpha, \beta)$ $1 \leqslant \beta < \alpha$	$U_i(\alpha)$	$U_0(\alpha)$
$\int_U \psi_{\boldsymbol{a}}$	1	0	-1 *if* $\alpha = 1$; 0 *else*	$-1 + r$ *if* $\alpha = 1$; 0 *else.*

PROOF. One can assume that $\boldsymbol{a} = (0,1)$. Then

$$\int_{U_i(\alpha,\beta)} \psi_{\boldsymbol{a}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = p^{2\alpha-\beta}\frac{p-1}{p}\int_{\mathbf{Z}_p^\times} e^{2\pi i u/p^\alpha}\mathrm{d}u$$
$$\int_{U_i(\alpha)} \psi_{\boldsymbol{a}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = p^{\alpha}\int_{\mathbf{Z}_p^\times} e^{2\pi i u/p^\alpha}\mathrm{d}u.$$

Since for $\lambda \in \mathbf{N}$

$$\int_{\mathbf{Z}_p} e^{2\pi i u/p^\lambda}\mathrm{d}u = \begin{cases} 1 & \text{if } \lambda = 0 \\ 0 & \text{if } \lambda \geqslant 1. \end{cases}$$

we have

$$\begin{aligned}\int_{\mathbf{Z}_p^\times} e^{2\pi i u/p^\alpha}\mathrm{d}u &= \int_{\mathbf{Z}_p} e^{2\pi i u/p^\alpha}\mathrm{d}u - \int_{p\mathbf{Z}_p} e^{2\pi i u/p^\alpha}\mathrm{d}u \\ &= \int_{\mathbf{Z}_p} e^{2\pi i u/p^\alpha}\mathrm{d}u - p^{-1}\int_{\mathbf{Z}_p} e^{2\pi i u/p^{\alpha-1}}\mathrm{d}u = \begin{cases} -1/p & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha \geqslant 2 \end{cases}\end{aligned}$$

hence colums two and three.

Since p does not divide $\boldsymbol{a}$, one can perform a change of variables to get

$$\int_{p^{-\alpha}(\mathbf{Z}_p^\times)^2} \psi_{\boldsymbol{a}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \begin{cases} -1 & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha \geqslant 2. \end{cases}$$

Substracting the integral coming from $U_i(\alpha',\beta')$ (which is zero) and $U_i(\alpha')$, one gets

$$\int_{U_0(\alpha)} \psi_{\boldsymbol{a}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \begin{cases} -1+r & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha \geqslant 2. \end{cases}$$

$\square$

LEMMA 2.5. *Let $\boldsymbol{s} \in \mathbf{C}^{s+1}$ such that $\Re(s_0) > 2$ and $\Re(s_i) > 1$ for $i \in \{1, ..., r\}$. Then for every generic character $\boldsymbol{a}$ and $p \notin S(\boldsymbol{a})$,*

$$\widehat{H}_p(\boldsymbol{s};\psi_{\boldsymbol{a}}) = 1 - \sum_{i=1}^r p^{-s_i} + (r-1)p^{-s_0}.$$

In particular, the Euler product

$$\prod_{p\notin S(\boldsymbol{a})\cup\{\infty\}} \widehat{H}_p(\boldsymbol{s};\psi_{\boldsymbol{a}})$$

converges absolutely to a holomorphic function on

$$\Omega(\varepsilon) = \{\boldsymbol{s} \in \mathbf{C}^{r+1} \mid \Re(s_0) > 5/2+\varepsilon \text{ and } \Re(s_i) > 3/2+\varepsilon \text{ for } i = 1, ..., r\}$$

for every $\varepsilon > 0$.

In particular,

$$\widehat{H}_p(\omega_X^{-s},\psi_{\boldsymbol{a}}) = 1 - rp^{-2s} + (r-1)p^{-3s}.$$

2.5.3. *Special characters.*

Lemma 2.6. *Let $\boldsymbol{a}$ be a character which is special for ℓ_i, $p \notin S(\boldsymbol{a})$ and $j \neq i$. We have the following table.*

U	$U(0)$	$U_i(\alpha,\beta)$	$U_i(\alpha)$
$\int_U \psi_{\boldsymbol{a}}$	1	$-p^{\alpha}\frac{p-1}{p}$ *if* $\beta = \alpha - 1$; 0 *else*	$p^{\alpha}\frac{p-1}{p}$

$U_j(\alpha,\beta)$ $j \neq i$	$U_j(\alpha)$ $j \neq i, 0$	$U_0(\alpha)$
0	-1 *if* $\alpha = 1$; 0 *else*	$-1 + r - p$ *if* $\alpha = 1$; 0 *else.*

Lemma 2.7. *Let $\boldsymbol{s} \in \mathbf{C}^{s+1}$ such that $\Re(s_0) > 2$ and $\Re(s_j) > 1$ for $j \in \{1, ..., r\}$. Then for every special character $\boldsymbol{a}$, being special for ℓ_i, and $p \notin S(\boldsymbol{a})$,*

$$\widehat{H}_p(\boldsymbol{s}; \psi_{\boldsymbol{a}}) = 1 - \sum_{j \neq i} p^{-s_j} + (r - p - 1)p^{-s_0} + \frac{(p-1)(1-p^{1-s_0})}{p^{s_i} - p}.$$

Proof. Remark that if $\boldsymbol{a}$ is special for ℓ_i, then it behaves as if it was generic for $j \neq i$, as our tables show. So there is only two new integrals to take into account: the one over $U_i(\alpha, \alpha - 1)$ and the one over $U_i(\alpha)$, while there only a change of value on $U_0(1)$ to take into account.

It gives

$$\widehat{H}_p(\boldsymbol{s}; \psi_{\boldsymbol{a}}) = \int_{U(0)} + \sum_{\alpha=1}^{\infty} \int_{U(\alpha,\alpha-1)} + \sum_{\alpha=1}^{\infty} \int_{U_i(\alpha)} + \sum_{j \neq i} \int_{U_j(1)}$$

$\dots$

$\square$

2.6. Bad reduction. Since we have to deal with a finite number of places, general estimates are sufficient.

Lemma 2.8. *Let $\boldsymbol{a} \in \mathbf{Z}^2$*
Then, there exist constants C' and $\eta > 0$ such that

$$\prod_{p \in S(\boldsymbol{a})} \leqslant C'(1 + \|\boldsymbol{a}\|)^{\eta}.$$

In particular, we obtain a uniform bound for any $\kappa > 0$

$$|\widehat{H}(\boldsymbol{s}, \psi_{\boldsymbol{a}})| \leqslant C(\kappa) \frac{(1 + \|\boldsymbol{s}\|)^{\nu(\eta)}}{(1 + \|\boldsymbol{a}\|)^{\kappa}}.$$

2.7. Conclusion. Recall our decomposition

$$\sum_{\boldsymbol{a} \in \mathbf{Z}^2} \widehat{H}(\boldsymbol{s}, \psi_{\boldsymbol{a}}) = \underbrace{\widehat{H}(\boldsymbol{s}, \psi_{\mathbf{0}})}_{\mathcal{Z}_0(\boldsymbol{s})} + \underbrace{\sum_{\substack{\boldsymbol{a} \in \mathbf{Z}^2 \setminus \{\mathbf{0}\} \\ \boldsymbol{a} \text{ generic}}} \widehat{H}(\boldsymbol{s}, \psi_{\boldsymbol{a}})}_{\mathcal{Z}_{\text{gen}}(\boldsymbol{s})} + \sum_{i=1}^{r} \underbrace{\sum_{\substack{\boldsymbol{a} \in \mathbf{Z}^2 \setminus \{\mathbf{0}\} \\ \boldsymbol{a} \text{ special for } \ell_i}} \widehat{H}(\boldsymbol{s}, \psi_{\boldsymbol{a}})}_{\mathcal{Z}_i(\boldsymbol{s})}.$$

Let us focus first on the main term $\mathcal{Z}_0(\boldsymbol{s})$. We can rewrite the local factor as

$$\begin{aligned}\widehat{H}_p(\boldsymbol{s},\psi_{\mathbf{0}}) &= \left(1+p^{2-s_0}\right)\prod_{i=1}^{r}(1+p^{1-s_i})(1+O(p^{-(1+\varepsilon)}))\\ &= \left(\frac{1}{1-p^{2-s_0}}\prod_{i=1}^{r}\frac{1}{1-p^{2-s_0}}\right)\times\left(1-p^{4-2s_0}\right)\prod_{i=1}^{r}(1-p^{2-2s_i})(1+O(p^{-(1+\varepsilon)}))\end{aligned}$$

thus

$$\mathcal{Z}_0(\boldsymbol{s}) = h_0(\boldsymbol{s})\zeta(s_0-2)\zeta(s_1-1)...\zeta(s_r-1)$$

where $h_0(\boldsymbol{s})$ is the product of an absolutely convergent Euler product that does not vanish on $\Omega(\varepsilon)\cap\mathbf{R}^{r+1}$ with

$$h(3,2,...,2)\neq 0$$

and

$$\Omega(\varepsilon) = \{\boldsymbol{s}\in\mathbf{C}^{r+1} \mid \Re(s_0) > 5/2+\varepsilon \text{ and } \Re(s_i) > 3/2+\varepsilon \text{ for } i=1,...,r\}$$

for every $\varepsilon > 0$.

3. Tamagawa numbers of classical algebraic groups

Special linear groups, symplectic group, special orthogonal groups.

Definition 3.1 (Tamagawa number of an algebraic group). Let G be

3.1. Introduction: Siegel's formula as an Adelic volume. Let q be a quadratic form given by an $n\times n$ positive definite symmetric integral matrix A (where $n\geqslant 3$).

The equation

$${}^tXAX = A$$

defines an algebraic group $\mathrm{SO}(q)$ over $\mathrm{Spec}(\mathbf{Z})$ of dimension $\frac{1}{2}n(n-1)$.

$$\alpha_p(q) = \lim_{e\to\infty}\frac{\#\{X\in\mathcal{M}_n(\mathbf{Z}/p^e\mathbf{Z}) \mid {}^tX\overline{A}X=\overline{A}\}}{2p^{\frac{1}{2}n(n-1)e}} = \lim_{e\to\infty}\frac{\mathrm{SO}(q)(\mathbf{Z}/p^e\mathbf{Z})}{2p^{\frac{1}{2}n(n-1)e}}.$$

One should recognize the p-adic volume of $\mathrm{SO}(q)(\mathbf{Z}_p)$ (or a "singular series" in the sense of Hardy-Littlewood!). Then, the simple equality

$$\tau_{\mathbf{Q}}(\mathrm{SO}(q)) = 2$$

is equivalent to the mass formula

$$\sum_{q'}\frac{1}{\#\mathrm{O}_{q'}(\mathbf{Z})} = 2\prod_{v\in M_{\mathbf{Q}}}\alpha_v(q)^{-1}$$

where q' runs on the isomorphism classes of quadratic forms having same genus than q (meaning that they are equivalent over the local rings $\mathbf{Z}_p$ for each prime p and also equivalent over $\mathbf{R}$).

3.2. The Tamagawa number of the special linear group over the rationals.

THEOREM 3.1. *For any positive integer n,*

$$\tau_{\mathbf{Q}}(\mathrm{SL}_n) = \int_{\mathrm{SL}_n(\mathbf{Q})\backslash \mathrm{SL}_n(\mathbb{A}_{\mathbf{Q}})} \mathrm{d}\mu = 1$$

For $n = 2$, it boils down to showing that

$$\mu_p(\mathrm{SL}_2(\mathbf{Z}_p)) = \frac{\# \mathrm{SL}_2(\mathbf{F}_p)}{p^3} = \frac{p(p^2-1)}{p^3} = (1 - p^{-2}).$$

and

$$\mu_\infty(\mathrm{SL}_2(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{R})) = \zeta(2) = \pi^2/6.$$

Then, we will prove the general result by induction on n.

CHAPTER 4

Exercises

Chapter 1

Exercise 1. Let K be a valued field with absolute value $|.|$. Show that $|.|$ is non-archimedean if, and only if, the sequence $|n|,\ n \geq 1$ is bounded. (**Hint:** given $x \in K$ such that $|x| < 1$, estimate $|(1+x)^n|$.)

Exercise 2. Find a Cauchy sequence of rational numbers (for the p-adic absolute value) which has no limit in $\mathbf{Q}$ but does have a limit in some $\mathbf{Q}_p$.

Exercise 3. Let K be a non-Archimedean local field. Show that the topology of K is totally discontinuous: the only non-empty connected subsets are singletons. *Hint:* show that if two disks intersect, one must be contained in the other.

Exercise 4. Let k be a field and $K = k((t))$ the field of Laurent series

$$\sum_{n \in \mathbf{Z}} a_n T^n$$

(where $a_n = 0$ for $n << 0$) having coefficients in k. If $a \neq 0$ set

$$|a| = e^{-n}$$

where n is the smallest integer such that $a_n \neq 0$ and $e = \exp(1)$.

(1) Show that it defines an ultrametric absolute value on K (which is called the t-adic absolute value with base e) and that K is complete with respect to this absolute value.
(2) Show that K is locally compact if and only if k is finite.

Exercise 5. Let R be the valuation ring of a non-archimedean local field. Show that the topology induced by the completion of $\mathbf{Q}$ (or $\mathbf{F}_p(T)$) and the profinite topology are equivalent.

Exercise 6. The goal of this exercise is to classify compact analytic manifolds over a **non-archimedean** local field K.

(1) Show that open disks in K^d are also closed. Show also that, if two open disks in K^d intersect, then one must be contained in the other.
(2) Show that any compact analytic manifold over K is isomorphic to a disjoint union of unit disks in K^d.
(3) Let q be the cardinality of the residue field of K. Show that any compact analytic manifold over K is the union of $m \in \{1, \ldots, q-1\}$ unit disks.

Exercise 7. Let K be a complete valued field and X be a K-scheme of finite type.

(1) If X is separated, show that $X(K)$ is Hausdorff: the diagonal immersion of topological spaces $X(K) \to X(K) \times X(K)$ is closed.

(2) Assume that K is locally compact. If X is proper, show that $X(K)$ is compact as follows.
  (a) First show it for $\mathbf{P}^n(K)$.
  (b) Then assume that X is projective and use the previous case.
  (c) When X is proper, Chow's lemma says that there exists a proper birational morphism $p : Y \to X$ such that Y is projective. Use this lemma to conclude.

(3) From now on, assume that K is **non-archimedean**, with valuation ring R and residue field k. Let $\mathscr{X}$ be a separated R-scheme of finite type such that $X_K = X$. For the next questions, show that one can assume that $\mathscr{X}$ is a closed subscheme of $\mathbf{A}^n_R$ for some $n \geqslant 0$.

(4) Show that the map $\mathscr{X}(R) \to X(K)$ is injective.

(5) Show that $\mathscr{X}(R)$ is open and closed in $X(K)$.

(6) Assume that K is a local field. Show that $\mathscr{X}(R)$ is a compact subset of $X(K)$.

(7) Show that the reduction map
$$\mathscr{X}(R) \to \mathscr{X}(k)$$
is anticontinuous: the preimage of any open subset in $\mathscr{X}(k)$ is closed in $\mathscr{X}(R)$.

(8) Show that for every $\tilde{x} \in \mathscr{X}(k)$ the fibre $\pi^{-1}(\tilde{x})$ is open and closed in $\mathscr{X}(R)$.

Exercise 8 (**to hand in**). The goal of this exercise is to build a canonical measure on analytic manifolds associated to singular schemes.

Let R be the valuation ring of a local field K. Consider X a (possibly singular) scheme of dimension d over R, with smooth locus X^{sm} (relative to R). We define the analytic manifold $X^{\natural} := X^{\mathrm{sm}}(K) \cap X(R)$.

(1) Suppose that there exists a line bundle over X which restricts to $\Omega^d_{X/R}$ over X^{sm}. Define a canonical measure on $X^{\natural}$.

Let C be a complex (possibly singular) algebraic curve. Consider the normalisation $\nu : \tilde{C} \to C$. The canonical sheaf Ω_C is defined as follows: over an open $U \subseteq C$, $\Omega_C(U)$ is the space of meromorphic 1-forms η on $\nu^{-1}(U)$ such that, for all $p \in U$ and all $f \in \mathcal{O}_{C,p}$, we have:
$$\sum_{\nu(q)=p} \operatorname{Res}(\nu^* f \cdot \eta) = 0.$$
A curve is called Gorenstein if Ω_C is a line bundle. In that case, we consider a spreading out of C over R, such that Ω_C spreads out to a line bundle.

(2) Show that the cusp ($C = \{y^2 - x^3 = 0\}$) has infinite canonical volume. Recall that the normalisation of the cusp is $\nu : \mathbb{A}^1 \to C,\ t \mapsto (t^2, t^3)$. (You may assume without proof that the cusp is Gorenstein.)

(3) Show that a Gorenstein curve has finite canonical volume if, and only if, it is smooth.

In words, for Gorenstein singular curves, gauge forms around singular points must have a pole, so that the canonical measure diverges.

Chapter 2

Exercise 1. Compute the local zeta function of the cusp $(y^2 - x^3 = 0)$ by hand (without using Denef's formula).

Exercise 2. Compute an embedded resolution of the tacnode ($y^2 - x^4 = 0$), its local zeta function and check the monodromy conjecture (using A'Campo's formula).

Exercise 3. [**MV24**, Thm. B] Let K be a non-archimedean local field, with valuation ring R and residue field k ($q := \sharp k$). Let $f(x) = L_1(x) \cdots L_n(x)$ be the equation of a hyperplane arrangement in K^d, i.e. $L_i(x) \in R[x],\ 1 \leq i \leq n$ are the equations of distinct affine hyperplanes in K^d. The goal of this exercise is to compute the following multivariate local zeta function:

$$Z_f(s_1, \dots, s_n) = \int_{R^d} \prod_{i=1}^{d} |L_i(x)|^{s_i} dx.$$

Let $\mathcal{A}$ be the set of hyperplanes $H_i = \{L_i(x) = 0\} \subset K^d$ and $\mathcal{L}(\mathcal{A})$ be the set containing all non-empty intersections $H_{i_1} \cap \ldots \cap H_{i_k},\ k \geq 0$, ordered by reverse inclusion. We denote by $\hat{0} \in \mathcal{L}(\mathcal{A})$ the subspace K^d, seen as the intersection of zero hyperplanes. Throughout, we assume that $\mathcal{L}(\mathcal{A}) \simeq \mathcal{L}(\bar{\mathcal{A}})$, where $\bar{\mathcal{A}}$ is the set of hyperplanes obtained from $\mathcal{A}$ by base change to k. In particular, the local zeta function only depends on $\mathcal{A}$ and we denote it by $Z_{\mathcal{A}}$.

Let $\mu : \mathcal{L}(\mathcal{A}) \to \mathbb{Z}$ be the Möbius function defined recursively as follows:

- $\mu(K^d) = 1$;
- $\mu(V) = -\sum_{\substack{W \in \mathcal{L}(\mathcal{A}) \\ W \supsetneq V}} \mu(W)$, for all $V \in \mathcal{L}(\mathcal{A})$.

The characteristic polynomial of $\mathcal{A}$ is defined as:

$$\chi_{\mathcal{A}}(t) := \sum_{V \in \mathcal{L}(\mathcal{A})} \mu(V) t^{\dim(V)}.$$

(1) Show that:

$$\chi_{\mathcal{A}}(q) = \sharp \left(k^d \setminus \bigcup_{i=1}^{n} \bar{H}_i \right).$$

Given $V \in \mathcal{L}(\mathcal{A})$, we define two hyperplane arrangements:

- the subarrangement $\mathcal{A}_V := \{H \in \mathcal{A} \mid V \subset H\}$, contained in K^d;
- the restricted arrangement $\mathcal{A}^V := \left\{ V \cap H \;\middle|\; \substack{V \in \mathcal{A} \setminus \mathcal{A}_V \\ V \cap H \neq \emptyset} \right\}$, contained in V.

(2) Show that:

$$Z_{\mathcal{A}}(s_1, \dots, s_n) = \sum_{V \in \mathcal{L}(\mathcal{A})} q^{-d - \sum_{H_i \in \mathcal{A}_V} s_i} \chi_{\mathcal{A}^V}(q) \zeta_{\mathcal{A}_V}(s_i,\ H_i \in \mathcal{A}_V).$$

Let Δ be the set of chains (of arbitrary length) $F = \{V_1 \supsetneq V_2 \supsetneq \ldots \supsetneq V_k\}$ in $\mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$. Given $F \in \Delta$, we define:

$$\pi_F(t) = \prod_{l=0}^{k} (-t)^{\dim V_k} \cdot \chi_{\mathcal{A}_{V_k}^{V_{k+1}}}(-t^{-1}),$$

where by convention, $V_0 = K^d$ and $V_{k+1} = \emptyset$.

(3) Show that:

$$Z_{\mathcal{A}}(s_1, \dots, s_n) = \sum_{F \in \Delta} \pi_F(-q^{-1}) \cdot \prod_{V \in F} \frac{q^{-\operatorname{codim}(V) - \sum_{H_i \supset V} s_i}}{1 - q^{-\operatorname{codim}(V) - \sum_{H_i \supset V} s_i}}.$$

EXERCISE 4. [**Wys17**, Lem. 4.7] Let K be a non-archimedean local field, with valuation ring R, uniformiser ϖ, and residue field k ($q := \sharp k$). Consider $f_1, \ldots, f_n \in R[x_1, \ldots, x_d]$. The Poincaré series of $f_1, \ldots, f_n$ is defined as:

$$P_f(T) := \sum_{m \geq 0} N_{f,m} \cdot T^m,$$

where $N_{f,m}$ is the number of solutions to the equations $f_1(x) = \ldots = f_n(x) = 0$ in $(R/(\varpi^m))^d$.

(1) Show that:

$$P_f(q^{-d-s}) = \frac{1 - q^{-s} Z_f(s)}{1 - q^{-s}}.$$

(2) Suppose that $Z_f(s)$ has its closest pole to the origin at $q^{-s} = q^n$ and that this pole is simple. Show that the sequence $\{q^{-(d-n)m} \cdot N_{f,m}\}_{m \geq 1}$ converges and that:

$$\lim_{m \to +\infty} \left(q^{-(d-n)m} \cdot N_{f,m}\right) = \frac{-1}{q^n - 1} \cdot \mathrm{Res}_{q^n} Z_f.$$

EXERCISE 5. [**Igu00**, Thm. 11.7.2] Let $f \in \mathbf{Q}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d. The goal of this exercise is to prove a functional equation satisfied by $Z_f(s)$, due to Denef and Meuser.

Let us call $\mathbf{Q}_{p^e}$ the unique unramified extension of $\mathbf{Q}_p$ of degree e and $Z_f^{(e)}(s)$ the zeta function of f over $\mathbf{Q}_{p^e}$. Throughout the exercise, we assume that there exists a rational fraction $Z \in \bar{\mathbf{Q}}(u, v)$ such that $Z_f^{(e)}(s) = Z(q^{-e}, q^{-es})$. The functional equation proved by Denef and Meuser is:

$$Z(u^{-1}, v^{-1}) = v^d \cdot Z(u, v).$$

We admit that there exists an embedded resolution of singularities $h : Y \to \mathbb{A}^n_{\mathbf{C}}$ of $\{f = 0\} \subset \mathbb{A}^n_{\mathbf{C}}$, such that Y is endowed with an action of $\mathbf{C}^\times$ and h is equivariant with respect to the scaling action on $\mathbb{A}^n_{\mathbf{C}}$. We further assume that this resolution has good reduction modulo p.

(1) Set $q := p^e$. Show that:

$$Z_f^{(e)}(s) = \frac{1}{1 - q^{-(ds+n)}} \cdot \int_{\mathbf{Z}_q^n \setminus p\mathbf{Z}_q^n} |f(x)|^s dx.$$

Let E_i, $1 \leq i \leq t$ be the irreducible components of the total transform of $\{f = 0\}$ under h and $(N_i, \nu_i)_{1 \leq i \leq t}$. Say that $E_1, \ldots, E_{t'}$ are the components which are not contained in $h^{-1}(0)$. Let Y^* and E_i^*, $1 \leq i \leq t'$ be the quotients $Y/\mathbf{C}^*$ and $(E_i \setminus h^{-1}(0))/\mathbf{C}^\times$, for $1 \leq i \leq t'$. Finally, given $I \subset \{1, \ldots, t'\}$, let us call $E_I^* = \bigcap_{i \in I} E_i^* \setminus \bigcup_{i \notin I} E_i^*$. Note that E_I^* is smooth, projective, of dimension $n - 1 - \sharp I$ (if non-empty).

(2) Show that:

$$Z_f^{(e)}(s) = \frac{q^{ds}(q-1)}{q^{(ds+n)} - 1} \cdot \sum_{I \subset \{1, \ldots, t'\}} \sharp E_I^*(\mathbf{F}_q) \cdot \prod_{i \in I} \left(\frac{q-1}{q^{N_i s + \nu_i} - 1} - 1\right).$$

From the Weil conjectures, we know that, for each $I \subset \{1, \ldots, t'\}$, there exist algebraic numbers $\alpha_{I;i,j}$, where $1 \leq i \leq 2(n - 1 - \sharp I)$ and $1 \leq j \leq b_i := b_i(E_I^*)$, such that $b_i = b_{2(n-1-\sharp I)-i}$ and we have:

$$\left\{\frac{q^{n-1-\sharp I}}{\alpha_{I;i,j}},\ 1 \leq j \leq b_i\right\} = \left\{\alpha_{I;2(n-1-\sharp I)-i,j},\ 1 \leq j \leq b_{2(n-1-\sharp I)-i}\right\},$$

$$\sharp E_I^*(\mathbf{F}_q) = \sum_{i=0}^{2(n-1-\sharp I)} \sum_{j=1}^{b_i} (-1)^i \alpha_{I;i,j}^e.$$

(3) Show that:

$$Z(q, q^s) = q^{-sd} \cdot Z(q^{-1}, q^{-s})$$

for all $e \geq 1$ and $\mathrm{Re}(s) \geq 0$. Deduce from this Denef and Meuser's functional equation.

EXERCISE 6. [**Wys17**, Exmp. 4.15], (**to hand in**) The goal of this exercise is to compute the following local zeta function:

$$Z(s) := \int_{\mathbf{Z}_p^{2n}} |\mu(x,y)|^s dxdy,$$

where $\mu : \mathbf{Z}_p^{2n} \to \mathbf{Z}_p$ is the polynomial map defined by:

$$\mu(x,y) = x_1y_1 + \ldots + x_ny_n.$$

We will use p-adic Fourier transform with respect to the additive character:

$$\begin{array}{rccc} \psi : & \mathbf{Q}_p & \to & \mathbf{C}^\times \\ & x & \mapsto & \exp\left(\frac{2\pi i}{p} x\right), \end{array}$$

which factors through $\mathbf{Q}_p \twoheadrightarrow \mathbf{Q}_p/p\mathbf{Z}_p \simeq \mathbf{Z}\left[\frac{1}{p}\right]/p\mathbf{Z}$ (see Chapter 3 or [**Wys17**, §4.1]). We will denote by $\mathbf{1}_A$ the characteristic function of a subset $A \subset \mathbf{Q}_p^n$. Let $B_k := p^k \mathbf{Z}_p^{2n}$, where $k \in \mathbf{Z}$. You may freely use the following fact:

$$\mathcal{F}(\mathbf{1}_{B_k}) = p^{-2nk} \cdot \mathbf{1}_{B_{-k+1}}.$$

Note that our choice of character slightly differs from the choice made in Chapter 3. Hence the Fourier transform is only an involution up to a non-trivial normalisation factor.

(1) Show that for every compact measurable subset $A \subset \mathbf{Q}_p$, we have:

$$\int_{\mathbf{Z}_p^{2n}} \mathbf{1}_A(\mu(x,y)) dxdy = q \cdot \int_{\mathbf{Q}_p} \mathcal{F}(\mathbf{1}_A)(z) \frac{dz}{\max\{1, (q|z|)^n\}}.$$

By definition, this means that the Fourier transform of the measure $\mu_* dxdy$ (in the sense of distributions) on $\mathbf{Q}_p$ is given by $\frac{dz}{\max\{1,(q|z|)^n\}}$.

(2) Show that:

$$Z(s) = \frac{(q-1)(q^n-1)q^{2s}}{(q^{s+1}-1)(q^{s+n}-1)}.$$

Chapter 3

EXERCISE 1. Let G be a locally compact abelian topological group. Show that $\perp$ bijectively sends closed subgroups of G to closed subgroups of $\widehat{G}$.

EXERCISE 2. Show that if K_v is non-archimedean, then any Schwartz-Bruhat function is a finite linear combination of characteristic functions of open balls.

EXERCISE 3. Compute the Fourier transform of the characteristic function of a ball.

EXERCISE 4 (To be done once in your life but maybe not twice). We construct non-trivial characters for non-Archimedean local fields.

(1) Given any $x \in \mathbf{Q}_p$, let $n \in \mathbf{N}$ be the smallest nonnegative integer such that
$$p^n x \in \mathbf{Z}_p.$$
Let r be such that
$$r \equiv p^n x \bmod p^n.$$
Put
$$\psi_p(x) = \exp(2\pi i r/p^n)$$
for all $x \in \mathbf{Q}_p$ and show that $\psi_p : \mathbf{Q}_p \to S^1$ is a nontrivial unitary character, of conductor $\mathbf{Z}_p$, which identifies with the composite map
$$\mathbf{Q}_p \to \mathbf{Q}_p/\mathbf{Z}_p \simeq \mathbf{Z}[1/p]/\mathbf{Z} \hookrightarrow \mathbf{Q}/\mathbf{Z} \hookrightarrow \mathbf{R}/\mathbf{Z} \simeq S^1.$$

(2) Let K_∞ denote the completion of $K = \mathbf{F}_q(t)$ at the place t^{-1}, the "place at infinity" (you can assume that $q = p$ is prime if you want). Show that
$$K_\infty = \mathbf{F}_q((t^{-1})).$$
In other words, elements of K_∞ are uniquely represented by formal power series of the form
$$\sum_{n \leqslant r} a_n t^n$$
with $a_n \in \mathbf{F}_q$. Put
$$\psi_\infty(x) = \exp(2\pi i \mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(a_1)/p)$$
for $x \in K_\infty$. Show that ψ_∞ is a nontrivial unitary character of K_∞ of conductor $\mathbf{F}_q[[t^{-1}]]$.
N.B. In applications, the completion K_∞, even if it is non-Archimedean, plays the role of $\mathbf{R} = \mathbf{Q}_\infty$ for $\mathbf{F}_q(t)$.

(3) Let π be any irreducible polynomial inside $\mathbf{F}_q[t]$ and let K_π be the completion of K with respect to the prime (π), with residue field k.
   (a) Check that every element of K_π can uniquely written in the form
$$\sum_{n \geqslant r} a_n \pi^n$$
   for some integer r and the a_n's in k.
   (b) Put
$$\psi_\pi(x) = \exp\left(\frac{2\pi i\, \mathrm{tr}_{k/\mathbf{F}_p}(a_{-1})}{p}\right)$$
   for all $x \in K_\pi$ and show that it defines a nontrivial unitary character ψ_π of K_π.

(4) Find a uniform and canonical way to define ψ_∞ and ψ_π at the same time.

(5) Let F be any non-Archimedean local field. Recall that F is a finite separable extension of a local field $F_0 = \mathbf{Q}_p$, K_∞ or K_π as before, coming respectively with a character $\psi_0 = \psi_p$, ψ_∞ or ψ_π constructed above. Put
$$\psi(x) = \psi_0(\mathrm{tr}_{F/F_0}(x))$$
and show that ψ is a nontrivial unitary character of F.

EXERCISE 5 (**to hand in**). Let $f \in \mathbf{Z}[x_1, ..., x_n]$ be a homogeneous polynomial of degree d with $n \geqslant 3$ and

$$X \to \mathbf{P}^n = \mathrm{Proj}(\mathbf{Z}[x_0, ..., x_n])$$

the blow-up of the homogeneous ideal (x_0, f).

The polynomial f defines an hypersurface Z_f in $\mathbf{P}^{n-1} \simeq \{x_0 = 0\}$. We assume that $Z_{f,\mathbf{C}}$ is smooth, irreducible and does not contain any hyperplane.

Let $U \subset X$ be the inverse image in X of $\{x_0 \neq 0\} \simeq \mathbf{A}^n$.

The goal of this (probably too long) problem is to adapt what we have seen during the last lectures to show that

$$N_{U,\omega_X^{-1}}(B) = \#\{x \in U(\mathbf{Q}) \mid H_{\omega_X^{-1}}(x) \leqslant B\} \sim \theta(X) B \log(B)$$

when $B \to \infty$.
In this exercise we will not care very much about what is $\theta(X)$, the important fact is that it is positive.

We admit that the exceptional divisors D_1 and the strict transform D_0 of $\{x_0 = 0\}$ freely generate the Picard group of X, that

$$D(\boldsymbol{s}) = s_0[D_0] + s_1[D_1] \in \mathrm{Pic}(X)$$

is effective if and only if $(s_0, s_1) \in \mathbf{Z}_{\geqslant 0} \times \mathbf{Z}_{\geqslant 0}$ and that

$$[\omega_X^{-1}] = (n+1)[D_0] + n[D_1].$$

(1) The first step is to construct a height zeta function attached to the counting problem and apply the Poisson formula to it.
 (a) Show (using the general formula from the course) that the canonical metrics associated to D_1 are

$$\boldsymbol{x} \in \mathbf{Q}_p^n \mapsto \max\left(\frac{1}{\max(1, \|\boldsymbol{x}\|_p)}, \frac{|f(\boldsymbol{x})|_p}{\max(1, \|\boldsymbol{x}\|_p)^d}\right)$$

 at every prime p and the square-root of

$$\boldsymbol{x} \in \mathbf{R}^n \mapsto \frac{1}{1 + \sum_{j=1}^n |x_j|^2} + \frac{|f(\boldsymbol{x})|^2}{1 + \sum_{j=1}^n |x_j|^2}$$

 at the archimidean place. Deduce the local exponential height associated to D_1 (it's just the inverse of the metric).
 (b) Show similarly that the local exponential heights associated to D_0 are

$$H_{D_0,p}(\boldsymbol{x}) = \max(1, |f(\boldsymbol{x})|_p) = \frac{\max(1, |\boldsymbol{x}|_p)}{H_{D_1,p}(\boldsymbol{x})}$$

 and

$$H_{D_0,\infty}(\boldsymbol{x}) = \sqrt{1 + f(\boldsymbol{x})^2} = \frac{\sqrt{1 + \sum |x_j|^2}}{H_{D_1,p}(\boldsymbol{x})}.$$

 (c) For any place $v \in M_{\mathbf{Q}}$ and any $\boldsymbol{s} = (s_0, s_1) \in \mathbf{C}^2$ set

$$H_v(\boldsymbol{s}, \boldsymbol{x}_v) = H_{D_0,v}^{s_0}(\boldsymbol{x}_v) H_{D_1,v}^{s_1}(\boldsymbol{x}_v).$$

 Deduce a simple expression for $H_{\omega_X^{-1}} = H([\omega_X^{-1}], \cdot)$.

(d) We identify $\boldsymbol{a} \in \mathbf{Q}^n$ with the linear form

$$\langle\,\cdot\,, \boldsymbol{a}\rangle \in \mathrm{Hom}_{\mathrm{gp}}(\mathbf{G}_a^n, \mathbf{G}_a)(\mathbf{Q})$$

it defines on $\mathbf{G}_a^n$, and with $\psi_{\boldsymbol{a}} = \psi(\langle\,\cdot\,, \boldsymbol{a}\rangle)$.
Show that

$$Z(\boldsymbol{s}) = \sum_{\boldsymbol{a}\in\mathbf{Z}^2} \widehat{H}(\boldsymbol{s}, \psi_{\boldsymbol{a}})$$

where

$$\widehat{H}(\mathbf{s}, \psi_{\boldsymbol{a}}) = \int_{\mathbf{G}_a^n(\mathbb{A}_{\mathbf{Q}})} H(\boldsymbol{s}, \boldsymbol{x})^{-1}\psi_{\boldsymbol{a}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

(2) We compute the Fourier transform of the height function at the trivial character. By spreading out, there exists a minimal set of primes S such that $Z_f \subset \mathbf{P}^{n-1}_{\mathbf{Z}}$ is smooth above $\mathrm{Spec}(\mathbf{Z}[S^{-1}])$.

(a) We define a stratification of $\mathbf{Q}_p^n$ as follows.

	$U(0) = \mathbf{Z}_p^n$	$U_1(\alpha,\beta)$ $0<\beta<\alpha$	$U_1(\alpha)$ $\alpha \geqslant 1$	$U(\alpha)$ $\alpha \geqslant 1$
$\lVert\boldsymbol{x}\rVert$		p^{α}	p^{α}	p^{α}
$\lvert f(\boldsymbol{x})\rvert$		$p^{d\alpha-\beta}$	$p^{(d-1)\alpha}$	$p^{d\alpha}$
$H_{D_0,p}$				
$H_{D_1,p}$				

Complete the two missing lines and the missing column.

(b) For every $p \notin S$, we set

$$\tau_p(f) = \left(1 - \frac{1}{p}\right)\frac{\#Z_f(\mathbf{F}_p)}{p^{n-2}}.$$

For $p \notin S$, show that

(i)
$$\mathrm{Vol}(U_1(\alpha,\beta)) = \frac{p-1}{p}\tau_p(f)p^{n\alpha-\beta}.$$

(ii)
$$\mathrm{Vol}(U_1(\alpha)) = \tau_p(f)p^{(n-1)\alpha}.$$

(iii)
$$\mathrm{Vol}(U(\alpha)) = (1 - p^{-n} - p^{-1}\tau_p(f))p^{n\alpha}.$$

(iv)
$$\widehat{H}_p(\boldsymbol{s}, \psi_{\mathbf{0}}) = \frac{1-p^{-s_0}}{1-p^{n-s_0}} + \tau_p(f)\frac{p^{s_0-n} - p^{s_1-n}}{(p^{s_0-n}-1)(p^{s_1-n+1}-1)}.$$

(3) For $\boldsymbol{a} \neq 0$ and $0 \leqslant \alpha \leqslant \beta$, define

$$I(\alpha,\beta) = \int_{\substack{\lVert\boldsymbol{x}\rVert_p = p^{\alpha}\\ \lvert f(\boldsymbol{x})\rvert_p \leqslant p^{d\alpha-\beta}}} \psi_{\boldsymbol{a}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

Let $S(\boldsymbol{a})$ be the union of S and the primes p such that $\boldsymbol{a} \in p\mathbf{Z}^n$. In what follows we assume that $p \notin S(\boldsymbol{a})$.

(a) Let $t \in \mathbf{Q}_p$. Show that

$$\frac{\int_{\mathbf{Z}_p^\times} \psi(tu)\mathrm{d}u}{\mathrm{Vol}(\mathbf{Z}_p^\times)} = \begin{cases} 1 & \text{if } t \in \mathbf{Z}_p \\ -\frac{1}{p-1} & \text{if } v_p(t) = -1 \\ 0 & \text{if } v_p(t) \leqslant -2 \end{cases}$$

(b) Using a change of variables, show that for any fixed $u \in \mathbf{Z}_p^\times$,

$$I(\alpha, \beta) = p^{n\alpha} \int_{\substack{\|\boldsymbol{y}\|=1 \\ |f(\boldsymbol{y})| \leqslant p^{-\beta}}} \psi\left(p^{-\alpha}\langle \boldsymbol{a}, \boldsymbol{y}\rangle u\right) \mathrm{d}\boldsymbol{y}.$$

(c) Integrate over $u \in \mathbf{Z}_p^\times$ and use the mean value computed before to show that

$$\begin{aligned} &I(\alpha, \beta) \\ &= p^{n\alpha}\left(\frac{p}{p-1}\mathrm{Vol}(\|\boldsymbol{x}\| = 1 \mid p^\beta | f(\boldsymbol{x}) \text{ and } p^\alpha | \langle \boldsymbol{a}, \boldsymbol{x}\rangle)\right. \\ &\left. - \frac{1}{p-1}\mathrm{Vol}(\|\boldsymbol{x}\| = 1 \mid p^\beta | f(\boldsymbol{x}) \text{ and } p^{\alpha-1} | \langle \boldsymbol{a}, \boldsymbol{x}\rangle)\right) \end{aligned}$$

(d) If $1 \leqslant \beta \leqslant \alpha$ show that

$$\mathrm{Vol}(\|\boldsymbol{x}\| = 1 \mid p^\beta | f(\boldsymbol{x}) \text{ and } p^\alpha | \langle \boldsymbol{a}, \boldsymbol{x}\rangle) = p^{-\alpha} p^{(2-n)\beta}\left(1 - \frac{1}{p}\right) \# Z_{f,\boldsymbol{a}}(\mathbf{Z}/p^\beta \mathbf{Z})$$

where

$$Z_{f,\boldsymbol{a}} = Z_f \cap \{\boldsymbol{x} \mid \langle \boldsymbol{x}, \boldsymbol{a}\rangle = 0\}.$$

(e) If $\alpha > 0$ show that

$$\begin{aligned} &\mathrm{Vol}(\|\boldsymbol{x}\| = 1 \mid p^\alpha | \langle \boldsymbol{a}, \boldsymbol{x}\rangle) \\ &= (1 - p^{1-n})p^{-\alpha}. \end{aligned}$$

(f) Deduce that $I(\alpha, \beta) = 0$ as soon as $1 \leqslant \beta < \alpha$ and compute the remaining values of I.

(g) Deduce that

$$\begin{aligned} \widehat{H}(\boldsymbol{s}, \psi_{\boldsymbol{a}}) =& 1 - p^{s_0} + (p^{s_1 - s_0} - 1)p^{s_1} \# Z_f(\mathbf{F}_p) \\ &- (p^{s_1-s_0} - 1)(1 - p^{n-s_1-2}) \sum_{\alpha=1}^{\infty} p^{-\alpha(s_1-1)} \# Z_{f,\boldsymbol{a}}(\mathbf{Z}/p^\alpha \mathbf{Z}). \end{aligned} \tag{1}$$

We admit that there exists a constant $C > 0$ such that for all α and all $p \notin S(\boldsymbol{a})$

$$\# Z_{f,\boldsymbol{a}}(\mathbf{Z}/p^\alpha \mathbf{Z}) \leqslant C\left(p^{(n-3)\alpha} + p^{(n-2)(\alpha-1)}\right).$$

(4) Fix $\varepsilon > 0$ and set

$$\Omega(\varepsilon) = \left\{\boldsymbol{s} \in \mathbf{C}^2 \;\middle|\; \Re(s_0) > n + \frac{1}{2} + \varepsilon \text{ and } \Re(s_1) > n - \frac{1}{2} + \varepsilon\right\}.$$

(a) Using the previous computations, show that

$$\prod_{p<\infty} \widehat{H}_p(\boldsymbol{s},\psi_{\mathbf{0}})(1-p^{n-s_0})(1-p^{n-1-s_1})$$

converges whenever $\Re(s_0) > n+1$ and $\Re(s_1) > n$.

(b) Show that

$$\widehat{H}(\boldsymbol{s},\psi_{\mathbf{0}}) = \zeta(s_0-n)\zeta(s_1-n+1)\widehat{H}_\infty(\boldsymbol{s},\psi_{\mathbf{0}}) \prod_{p<\infty} \widehat{H}_p(\boldsymbol{s},\psi_{\mathbf{0}})(1-p^{n-s_0})(1-p^{n-1-s_1}).$$

(c) Deduce that there exists a holomorphic function g on Ω having polynomial growth in vertical strips and such that

$$\widehat{H}(\boldsymbol{s},\psi_{\mathbf{0}}) = \frac{g(\boldsymbol{s})}{(s_0-n-1)(s_1-n)}$$

and compute explicitly $g(n+1,n)$ in terms of a product involving the local factors $\tau_p(f)$ (possibly excluding finitely many places).

(d) The goal now is to control the coefficients of $\widehat{H}(\boldsymbol{s},\psi_{\boldsymbol{a}})$ for $\boldsymbol{a} \neq \mathbf{0}$. The following bound is really typical: show that there exists a constant $C > 0$ such that for all $\boldsymbol{a} \neq 0$ all $p \notin S(\boldsymbol{a})$ and all $\boldsymbol{s} \in \Omega$

$$|\widehat{H}_p(\boldsymbol{s},\psi_{\boldsymbol{a}} - 1| \leqslant Cp^{-3/2}.$$

(e) We omit the final step: one can show that for each $\boldsymbol{a} \neq \mathbf{0}$, $\widehat{H}(\boldsymbol{s},\psi_{\boldsymbol{a}})$ is holomorphic on Ω and that there are absolute constants $C' > 0$ and η such that

$$|\widehat{H}(\boldsymbol{s},\psi_{\boldsymbol{a}})| \leqslant C(1+\|\Im(\boldsymbol{s})\|)^\eta(1+\|a\|)^{-n-1}.$$

(5) Deduce that there exists a holomorphic function G on $\Omega(0)$ such that

$$Z(\boldsymbol{s}) = \frac{G(\boldsymbol{s})}{(s_0-n-1)(s_1-n)}.$$

Taking $\boldsymbol{s} = s(n+1,n)$ and using the following Tauberian theorem, conclude that $N_{U,\omega_X^{-1}}(B)$ admits the expected asymptotic.

THEOREM 0.1. *Let $(\lambda_n)_{n\in}$ be an increasing sequence of positive real numbers, $(c_n)_{n\in\mathbf{N}}$ a sequence of non-negative real numbers, and f the Dirichel series*

$$f(s) = \sum_{n=0}^{\infty} c_n\lambda_n^{-s}.$$

Assume that

- *f converges on some half-plane $\Re(s) > a > 0$*
- *f admits a meromorphic continuation to some half-plane $\Re(s) > a-\varepsilon > 0$*
- *on this domain, f admits a unique pole at $s = a$ of order $b \in \mathbf{N}$. Let $\theta = \lim_a f(s)(s-a)^b > 0$.*

Then

$$N(B) = \sum_{\lambda_n \leqslant B} c_n \sim \frac{\theta}{a(b-1)!} B^a \log(B)^{b-1}$$

as $B \to \infty$.

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